In a game, a player begins with 6 chips and has the goal of obtaining 16 chips through a series of bets. In each bet, the player wagers an amount $s$ and either wins or loses, with the current fortune changing increasing or decreasing by $s$ accordingly. It is not allowed to bet more than the current fortune or to bet so that the fortune would exceed 16 if the bet is won. For some betting strategies, the amount bet can vary depending on the current fortune. Each individual bet is won with probability $\theta$ where $\theta \neq 0.5$ and lost with probability $1 - \theta$. The player is said to win if he achieves a fortune of 16 chips.

1. (4 points) If the player bets one chip per bet repeatedly until he either goes broke or achieves a fortune of 16 chips, what is the probability that he wins in terms of $\theta$?

Solution: Apply gambler’s ruin with $a = 6$ and $b = 10$.

\[
\frac{1 - \left(\frac{1-\theta}{\theta}\right)^6}{1 - \left(\frac{1-\theta}{\theta}\right)^{16}}
\]

2. (4 points) If the player bets two chips per bet repeatedly until he either goes broke or achieves a fortune of 16 chips, what is the probability that he wins in terms of $\theta$?

Solution: Apply gambler’s ruin with $a = 6/2 = 3$ and $b = 10/2 = 5$.

\[
\frac{1 - \left(\frac{1-\theta}{\theta}\right)^3}{1 - \left(\frac{1-\theta}{\theta}\right)^8}
\]

3. (4 points) Suppose that the player adopts a bold strategy in which he bets everything if the current fortune is less than or equal to 8 and bets just what is needed to get to 16 in a single bet if the current fortune exceeds 8. Find the probability of winning in terms of $\theta$ under the bold strategy.

Solution: We could draw a probability tree to track the possible paths with this strategy. Let $b(x)$ represent the probability of winning under the bold strategy.

\[
\begin{align*}
b(6) &= \theta b(12) \\
b(12) &= \theta + (1 - \theta)b(8) \\
b(8) &= \theta
\end{align*}
\]

By sequential substitution,

\[
b(6) = \theta^2 + \theta^2(1 - \theta) = \theta^2(2 - \theta)
\]

4. (4 points) Suppose that the player adopts a power-of-two strategy in which he bets just enough to get his current fortune to the next integer power of 2. For example, the first bet would be of size 2 to potentially raise the fortune from 6 to $8 = 2^3$ in a single bet. The relevant powers of two for this strategy are 1, 2, 4, 8, 16. Find the probability of winning in terms of $\theta$ under the power-of-two strategy.

Solution: First bet 2. The resulting fortune is either 8 (if the bet wins) or 4 (if the bet loses). In either case, the fortune is a power of 2 and so all subsequent bets are for all or nothing. Let $p(x)$ be the probability of winning using this strategy.

\[
\begin{align*}
p(6) &= \theta p(8) + (1 - \theta)p(4) \\
p(8) &= \theta \\
p(4) &= \theta p(8)
\end{align*}
\]
By substitution,

\[ p(6) = \theta^2 + \theta^2(1 - \theta) = \theta^2(2 - \theta) \, . \]

Notice that the probability of winning under this strategy is equal to the probability of winning under the bold strategy.

5. (4 points) Evaluate the probability of winning for each of the above strategies for \( \theta = 1/3 \) and for \( \theta = 2/3 \). Which strategy or strategies are best in each case?

Solution: The following table displays these probabilities.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>( \theta = 1/3 )</th>
<th>( \theta = 2/3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bet 1</td>
<td>[ \frac{1 - 2^6}{1 - 2^{16}} = \frac{63}{65535} \approx 0.0009613 ]</td>
<td>[ \frac{1 - (1/2)^8}{1 - (1/2)^{16}} = \frac{64512}{65535} \approx 0.9844 ]</td>
</tr>
<tr>
<td>Bet 2</td>
<td>[ \frac{1 - 2^3}{1 - 2^8} = \frac{7}{255} \approx 0.02745 ]</td>
<td>[ \frac{1 - (1/2)^3}{1 - (1/2)^8} = \frac{224}{255} \approx 0.8784 ]</td>
</tr>
<tr>
<td>Bold/Power of 2</td>
<td>( (1/3)^2(2 - 1/3) = 5/27 \approx 0.1852 )</td>
<td>( (2/3)^2(2 - 2/3) = 16/27 \approx 0.5926 )</td>
</tr>
</tbody>
</table>

With \( \theta = 1/3 \), the bold and power-of-two strategies are equally good. With \( \theta = 2/3 \), the bet one strategy is the best.

For strategies of this type where one makes a sequence of bets and either wins or loses the amount bet, when \( \theta < 0.5 \), the bold strategy will always be at least as good as any other strategy. The only other strategies that can be equally good are of the form that some bets may raise the ratio of the fortune over the target to an integer power of 0.5. Here, since the target is itself a power of 2, any strategy with either bold bets or bets that could raise the fortune to a power of 2 will be the best possible. If the target were 100, bold bets or bets that raise the fortune to 25 or 50 when won are ideal. (If fractional bets are allowed, then 12.5, 6.25, and so on are also valid targets for individual bets.)

The take home message is if the game is unfavorable, it is best not to play. But if you must gamble until the end, it is best to be bold. If the game is favorable, it is good to make small bets for as long as possible.

There are alternative criteria for finding optimal betting strategies, for example by trying to maximize the expected gain per bet while minimizing the probability of going broke.