Assignment #13 — Due Wednesday, December 3, 2008, by 5:00 P.M.

Turn in homework in lecture, discussion, or your TA’s mailbox. Indicate the discussion section in which you expect to attend to pick up this assignment on the assignment.

311: Monday 1:20–2:10
312: Monday 12:05–12:55

1. (Review) Suppose that $X_1, X_2, \ldots \sim \text{i.i.d. Geometric}(0.3)$. Recall that the sum $X_1 + X_2 + \cdots + X_n$ has a Negative-Binomial($n, 0.3$) distribution.

   (a) Use R and `pnbinom()` to compute numerically the exact probability $P(X_1 + X_2 + \cdots + X_{30} > 99.5)$.
   (Note that `pnbinom(x,r,theta)` evaluates the cdf of negative binomial distribution at $x$ for parameters $r$ and $\theta$.)

   (b) Apply the central limit theorem to use a normal approximation and estimate this probability.

2. (Review) Suppose that $X_1, X_2, \ldots \sim \text{i.i.d. Poisson}(7/3)$. Recall that $X_1 + X_2 + \cdots + X_n \sim \text{Poisson}(7n/3)$.

   (a) Use R and `ppois()` to compute numerically the exact probability $P(X_1 + X_2 + \cdots + X_{30} > 85.5)$.

   (b) Apply the central limit theorem to use a normal approximation and estimate this probability.

   (c) In comparing the approximations from this problem and the previous problem, which one is more accurate? Why might this be the case?

3. The $\chi^2$ (or chi-square) distribution is defined as the distribution that arises from the sum of squared independent standard normal random variables, $X = Z_1^2 + \cdots + Z_n^2$. This problem will step you through a derivation of the density function of the $\chi^2(n)$ distribution.

   (a) Let $n = 1$. To find the density of $X_1 = Z_1^2$ where $Z_1 \sim N(0, 1)$, note that the cumulative distribution function of $X$ is

   $$F_X(x) = P(X_1 \leq x) = P(Z_1^2 \leq x) = P(-\sqrt{x} \leq Z \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

   Use the chain rule for derivatives and the fact that the derivative of the cdf is the density to find the density of the $\chi^2(1)$ distribution.

   (b) Recall that $\Gamma(1/2) = \sqrt{\pi}$ and verify that the $\chi^2(1)$ distribution is equal to the Gamma(1/2, 1/2) distribution since both distributions share the same density function.

   (c) Recall that the moment generating function of the Gamma($\alpha, \lambda$) density is $m(s) = (\lambda/(\lambda - s))^\alpha$ for $s < \lambda$. Use this fact, the uniqueness of moment generating functions, and induction to show that if $Y_i \sim \Gamma(\alpha_i, \lambda)$ for $i = 1, \ldots, n$ and if these $\{Y_i\}$ are independent, then the sum $Y = \sum_{i=1}^n Y_i \sim \Gamma(\sum_{i=1}^n \alpha_i, \lambda)$ by showing that $Y$ has the same moment generating function as this gamma distribution.

   (d) Conclude that the $\chi^2(n)$ distribution is the same as the Gamma(n/2, 1/2) distribution.

4. A gambler will make a series of bets until she either raises her initial stake to $100 from $20 or loses everything. The probability of winning a bet is $\theta$. For this problem, calculate each probability for three separate values of $\theta$, $\theta_1 = 0.48$, $\theta_2 = 0.5$, and $\theta_3 = 0.52$. When a bet of size $b$ is won (or lost), the total the gambler holds increases (or decreases) by $b$. 

(a) Find the probability that she raises her fortune to $100 before going broke if she makes $1 bets for each value of $\theta$.

(b) If the gambler takes a bolder strategy and makes a series of $2$ bets, she can win (or lose) with fewer bets, on average. Find the probability of raising her fortune to $100$ before going broke using this strategy for each $\theta$.

(c) Repeat the previous calculations if the sizes of the bets are $5$, $10$, or $20$.

(d) Which of the previous strategies is the best for each value of $\theta$?

(e) A more advanced strategy would allow bets of different sizes depending on the current fortune. For example, a very bold gambler could bet everything if current fortune $a \leq 50$, and bet $100 - a$ if $a > 50$. By this strategy, the gambler either loses immediately or increases her fortune whenever her fortune is less than half the target and either wins immediately or decreases her fortune whenever the current fortune is more than half the target.

By conditioning on the result of the first bet using this very bold strategy, find a series of linear equations for the probability of eventually winning for possible initial fortunes $a$. (You need not do this for all $a$ between 0 and 100. It suffices to begin at $a = 20$ and follow the paths of what might happen. For example, if $g(a)$ is the probability of eventually winning, then $g(20) = \theta g(40) + (1 - \theta)g(0) = \theta g(40)$. Then find an equation for $g(40)$ by conditioning on what happens if she starts with that initial fortune.) Solve the equations to find the probability of eventually winning (given the initial fortune is $a = 20$) using this very bold strategy for each $\theta$.

(f) Suppose that bets must be in whole dollar increments. Try to find the best possible strategy for each $\theta$. You do not need to prove your result. Is it best to be bold or timid and how does this depend on the value of $\theta$?

(g) What do you notice about the probability of eventually winning for different strategies when $\theta = 1/2$?

Work to do, but not turn in.

• Read Chapter 11, sections 11.1 and 11.2.