Assignment #8 — Due Wednesday, October 29, 2008, by 5:00 P.M.

Turn in homework in lecture, discussion, or your TA’s mailbox. Indicate the discussion section in which you expect to attend to pick up this assignment on the assignment.

311: Monday 1:20–2:10  312: Monday 12:05–12:55

Some of the following problems require the program R. The motivation for using R is described briefly in your course syllabus. There are handouts which explains how to install R and how to use R for these problems on the course web page.

1. Suppose that \( X \sim \text{Gamma}(5, 1) \) and \( Y \sim \text{Gamma}(3, 1) \) and that \( X \) and \( Y \) are independent.
   
   (a) Find the joint density \( f_{X,Y} \). Draw a sketch of the portion of \( \mathbb{R}^2 \) where this joint density is positive.
   
   (b) If \( W = X + Y \) and \( Z = X / (X + Y) \), draw a sketch of the portion of \( \mathbb{R}^2 \) where the joint density \( f_{W,Z} \) is positive.
   
   (c) Find the joint density \( f_{W,Z} \).
   
   (d) Find the marginal densities of \( W \) and of \( Z \). If these are examples of named densities, identify them and the associated parameter values.
   
   (e) Are \( W \) and \( Z \) independent? Explain.

2. Suppose that random variable \( X = U / (\lambda (1 - U)) \) where \( U \sim \text{Uniform}(0, 1) \) and \( \lambda > 0 \) is a constant.
   
   (a) Find the cdf of \( X \) and take its derivative to determine its density.
   
   (b) Find the density of \( X \) using Theorem 2.6.4.

3. Let \( X \) have the Cauchy distribution with density \( 1 / (\pi (1 + x^2)) \) for \( -\infty < x < \infty \).
   
   (a) Find the cdf of \( X \).
   
   (b) Find the quantile function (inverse cdf) of \( X \).
   
   (c) Find a function \( X = g(U) \) where \( U \sim \text{Uniform}(0, 1) \) so that \( X \) has the Cauchy distribution.

4. A bucket contains 10 red balls and 15 blue balls. In each part, it is very useful to think of the random variable in the question as a sum of simpler random variables.
   
   (a) If a sample of four balls is selected with replacement and each ball in the bucket is equally likely to be selected at each draw, find the expected number of red balls in the sample.
   
   (b) If a sample of four balls is selected without replacement and each ball in the bucket is equally likely to be selected at each draw, find the expected number of red balls in the sample.
   
   (c) If balls are sampled repeatedly with replacement until the fourth blue ball is chosen, what is the expected number of red balls sampled?

5. Suppose that \( X \sim \text{Geometric}(\theta) \) on \( 0, 1, \ldots \) and let \( Y = \min\{X, 50\} \). Compute \( E(Y) \) and \( E(X - Y) \).

6. Suppose that \( X \) is a discrete random variable on \( k = 0, 1, 2, \ldots \) with \( P(X = k) = p_k \). Prove that \( \sum_{k=0}^{\infty} P(X > k) = E(X) \). (Hint: Write \( P(X > k) \) as a sum and then (carefully) change the order of summation.)
7. (Computer Problem). The program R has built in functions for computing probabilities, densities, and quantiles (the inverse cdf function) for many distributions. This problem asks you to use R to redo a problem from a previous assignment. For each problem, write down the R command and the numerical answer to four decimal places.

Let $Y \sim N(100, 5^2)$.

(a) Use `pnorm` to find $P(Y > 105)$;
(b) Use `pnorm` to find $P(Y > 92)$;
(c) Use `pnorm` to find $P(90 < Y < 110)$;
(d) Use `qnorm` to find the constant $y$ such that $P(Y > y) = 0.20$;
(e) Use `pnorm` to find $P(|Y - 100| > 112)$.
(f) Use `qnorm` to find the constant $y$ such that the constant $y$ (accurate to two decimal places) such that $P(|Y - 100| < y) = 0.9$.

Work to do, but not turn in.

- See the handout and load the program R onto your computer.
- Read Chapter 3 through section 3.4.