Assignment #7 (corrected and revised) — Due Wednesday, October 22, 2008, by 5:00 P.M.

Turn in homework in lecture, discussion, or your TA’s mailbox. Indicate the discussion section in which you expect to attend to pick up this assignment on the assignment.

311: Monday 1:20–2:10  312: Monday 12:05–12:55

1. Suppose that $X \sim \text{Binomial}(4, 0.1)$ and $Y$ is the number of 1’s in $X + 1$ rolls of a fair six-sided die.
   
   (a) Find the joint probability function for $X$ and $Y$.
   
   (b) Find the marginal distribution of $Y$.

2. Absolutely continuous random variables $X$ and $Y$ have joint density function
   
   \[ f_{X,Y}(x, y) = \begin{cases} 
   cx^2y^4 & \text{for } |x| + |y| < 2 \\
   0 & \text{otherwise}
   \end{cases} \]

   (a) Find the value of $c$.
   
   (b) Find the marginal density of $X$.
   
   (c) Find the marginal density of $Y$.
   
   (d) Find the conditional density of $Y$ given $X = 1$.
   
   (e) Find the conditional density of $X$ given $Y = 1$.

   Hint: Sketch the region where $f_{X,Y}(x, y) > 0$.

3. A fair coin is tossed five times. $X$ is the number of heads in the first three coin tosses. $Y$ is the number of heads in the last three coin tosses.

   Find the conditional probability distribution $p_{Y \mid X}(y \mid x)$ for each $x = 0, 1, 2, 3$.

4. Exercise 2.8.17. Let $X$ and $Y$ have the bivariate normal distribution as described in Example 2.7.9 on page 85, except let $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Prove that $X$ and $Y$ are independent if and only if $\rho = 0$. (In other words, show that if $\rho = 0$, then $f_{X,Y}(x, y) = f_X(x)f_Y(y) = \phi(x)\phi(y)$ and also show that if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, then it must be the case that $\rho = 0$. For the second part, you need to find the marginal densities of $X$ and $Y$.)

5. This exercise will step you through a derivation of the density of the order statistic $X_{(k)}$, the $k$th smallest value of an i.i.d. sample $X_1, \ldots, X_n$ where $X_i$ is an absolutely continuous random variable with cdf $F$ and density $f = F'$. The method will be to find the cdf of $X_{(k)}$ and take its derivative.

   (a) For a fixed value $x$ such that $0 < F(x) < 1$, let $I_i$ be an indicator random variable of the event $\{X_i \leq x\}$.

   What is the (named) distribution of $W = I_1 + \cdots + I_n$?

   (b) Write an expression that is a sum of probabilities for the event $\{W \geq k\}$.

   (c) Explain in words why the events $\{W \geq k\}$ and $\{X_{(k)} \leq x\}$ are identical. Thus, if $G_k(x) = \Pr(X_{(k)} \leq x)$ is the cdf of $X_{(k)}$, then $G_k(x)$ is equal to the expression you found in (b).

   (d) Let $g_k$ be the density of $X_{(k)}$ so that $g_k(x) = \frac{d}{dx}(G_k(x))$. Recalling that the derivative of a sum is the sum of the derivatives, express $g_k(x)$ as a sum. This sum should be of the form

   \[ \sum_{i=k}^{n} \left( a_i f(x)(F(x))^{i-1}(1 - F(x))^{n-i} - b_i f(x)(F(x))^i(1 - F(x))^{n-i-1} \right) \]

   for some constants (expressions that do not depend on $x$) $\{a_i\}$ and $\{b_i\}$. 
(e) This sum can be broken into a sum of positive terms (with \(a_i\)) and a sum of negative terms (with \(b_i\)). Notice that the first sum takes the form
\[
a_k f(x)(F(x))^{k-1}(1 - F(x))^{n-k} + a_{k+1} f(x)(F(x))^k (1 - F(x))^{n-k-1} + \cdots + a_n f(x)(F(x))^{n-1}(1 - F(x))^0
\]
and that the second sum takes the form
\[
-b_k f(x)(F(x))^k (1 - F(x))^{n-k-1} - \cdots - b_{n-1} f(x)(F(x))^{n-1}(1 - F(x))^0 - b_n f(x)(F(x))^n (1 - F(x))^{-1}
\]
so that the sums can be recombined by matching terms with common exponents and written as
\[
a_k f(x)(F(x))^{k-1}(1 - F(x))^{n-k} + \sum_{j=k}^{n-1} (a_{j+1} - b_j) f(x)(F(x))^j (1 - F(x))^{n-j-1} - b_n f(x)(F(x))^n (1 - F(x))^{-1}.
\]
Show that \(a_{j+1} = b_j\) and that \(b_n = 0\) so that the density of the order statistic \(X_{(k)}\) is
\[
g_k(x) = \frac{n!}{(n-k)!(k-1)!} f(x)(F(x))^{k-1}(1 - F(x))^{n-k}.
\]

(f) Consider the order statistic \(X_{(k)}\) from an i.i.d. sample where \(X_i \sim \text{Uniform}(0, 1)\). Use the result of the previous problem to show that \(X_{(k)} \sim \text{Beta}(k, n - k + 1)\). (Hint: \(\Gamma(n+1) = n!\) for nonnegative integer \(n\) and \(F\) and \(f\) are very simple for the Uniform distribution.)

Work to do, but not turn in.

• Read Chapter 3 through section 3.1.