Statistics 224   EXAM 3   Your Name ____________________

modified version of exam from Friday 11/30/07

The modifications reflect changes to expect on Exam 3 for 5/01/09

Professor Michael Iltis   (Lecture 2)

Discussion section (circle yours) :

section: 321 (3:30 pm M)
322 (2:25 pm M)
323 (4:35 pm M)

Problem	max points	points scored
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Do all 5 problems.

Rules :

1. No notes allowed
2. Standard hand calculator allowed
3. Numerical answers without supporting work (or rationale) may receive no points
4. Failure to follow rules may result in lost points

1. A physical anthropology study of the ability of individuals to walk in a straight line reported that in a sample of n = 20 randomly selected healthy men their cadence (which is the number of strides per second) data had a sample mean of

$$\bar{x} = .9255 \quad \text{and standard deviation} \quad s = .0809 \ .$$

A normal probability plot yielded substantial support to the assumption that the population distribution of cadence is (approximately) normal.

a) (5 points) Calculate a 95% confidence interval for population mean cadence.

When sampling from a normal population, $\bar{x}$ being a linear combination of normals is normal and after we standardize this to get a standard normal by subtracting its mean μ and dividing by the standard deviation $\sigma/\sqrt{n}$ of $\bar{x}$, replacing the unknown standard deviation σ by the sample standard deviation s defines a t random variable with n-1 = 19 degrees of freedom and with $\alpha = .05$ we solve the inequality (valid with probability $1-\alpha$)

$$-t_{\alpha/2,19} < t = \frac{\bar{X} - \mu}{S/\sqrt{20}} < t_{\alpha/2,19}$$

for the population mean μ yielding (before selecting the random sample)

$$P(\bar{X} - t_{\alpha/2} S/\sqrt{n} < \mu < \bar{X} + t_{\alpha/2} S/\sqrt{n}) = .95.$$

Once we replace the random variables by the particular random sample selected there is no longer any probability involved but we get the 95% confidence interval :

$$\bar{x} \pm t_{.025,19} s/\sqrt{n} = .92555 \pm 2.093(.0809)/\sqrt{20}$$

$$= .9255 \pm .03786$$

$$= [.88764, .96336] \ .$$

b) (5 points) **Interpret** this confidence interval. Your interpretation should answer the question of whether you know or in what sense the actual population mean cadence lies in the interval you found.

The **interpretation** is that we are 95% sure that μ lies in this interval in the sense that if we perform a large number of similar experiments each involving 20 randomly selected individuals (independent of past selections) that on average 95% of the time we would be correct in saying that μ lies in this interval. But for a given interval the actual population mean μ either lies in this confidence interval or it doesn't (no probability involved) and we don't know which holds.

1. c) (5 points) Calculate a 95% prediction interval for a future single value X_{21} :
Using the sample mean as a point estimate for X_{n+1} the prediction error $\bar{X}-X_{n+1}$ being a linear combination of normals will be normal, with mean

$E[\bar{X}-X_{n+1}]=\mu-\mu = 0$ and variance $V[\bar{X}-X_{n+1}]=V[\bar{X}]+V[X_{n+1}]=\frac{\sigma^2}{n}+\sigma^2=\sigma^2(1+\frac{1}{n})$.

Replacing the unknown standard deviation σ by the sample standard deviation s in the standard normal standardized prediction error yields a t random variable with n -1 degrees of freedom. Re-work the inequality $-t_{\alpha/2}< t = \frac{\bar{X}-X_{n+1}}{s\sqrt{1+\frac{1}{n}}}<t_{\alpha/2}$ with $\nu=n-1$ *d.f.* (with n = 20 here) to get a corresponding inequality (your $100(1-\alpha)\%$ prediction interval) for X_{21} .

The calculation proceeds very much like for the confidence interval. Solving for the future observation X_{21} we find that with probability $1-\alpha$

$$\bar{X}-t_{\alpha/2,n-1}S\sqrt{1+\frac{1}{n}}<X_{21}<\bar{X}+t_{\alpha/2,n-1}S\sqrt{1+\frac{1}{n}}$$

holds which is wider than the confidence interval by a factor of close to $\sqrt{n}$.
For the particular sample here we get the 95% prediction interval (P.I.) for X_{21}

$$\bar{x}\pm t_{.025,19}s\sqrt{1+\frac{1}{n}} = .92555\pm 2.093(.0809)\sqrt{1+\frac{1}{20}}$$
$$= .9255\pm .1735$$
$$= [.75199, 1.0990]$$

The interpretation of the statement that we are 95% sure that the future observation X_{21} lies in this interval is essentially the same as for being 95% sure that μ lies in the confidence interval . Once we observe the value x_{21} it either lies in the interval or it doesn't . Our future prediction made ahead of time will be correct on average 95% of the time if we perform many similar experiments.

d) (5 points) Calculate a 95% confidence interval for the true population standard deviation σ of men's cadence.
With probability 95% we have

$$\chi^2_{1-\alpha/2,n-1}=\chi^2_{.975,19}=8.906 < \chi^2=\frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2}=\chi^2_{.025,19}=32.852$$

or solving for σ with probability 95%

$$S\sqrt{\frac{19}{32.852}} < \sigma < S\sqrt{\frac{19}{8.906}} .$$

Once we replace the random variable S with the particular sample standard deviation s = .0809 there is no longer any probability but we get the confidence interval for σ

$$.06152 < \sigma < .11816$$

which is either true or false and we don't know which but we are 95% sure it is true in the same sense as for our confidence interval for μ .

2. The drying time for a certain type of paint test specimen is normally distributed with mean $\mu = 75$ minutes and and standard deviation $\sigma = 9$ minutes. Chemists have developed an additive which they believe will reduce the drying time. Only if conclusive evidence of $\mu < 75$ is found will the additive be adopted by the manufacturer. For a sample of size *n* = 25 drying times of paint specimens

a) (5 points) What are your null and alternative hypotheses here and why?

We test $H_0 : \mu = 75$ vs $H_a : \mu < 75$ since the alternative hypothesis should represent the claim which we are trying to establish.

b) (5 points) If the significance level of the test is $\alpha = .01$, determine the rejection region (i.e. the value of the test statistic $Z = \frac{\bar{X} - 75}{9/\sqrt{25}}$ below which $\mu = 75$ will be rejected in favor of $\mu < 75$). What should you conclude at level $\alpha = .01$ if we observe a sample mean of $\bar{x} = 72$?

Since the alternative hypothesis involves a value less than the null hypothesis value 75 we have a left one sided test. Evidence for such an alternative will be if the Z statistic is far to the left or in other words we will reject $H_0 : \mu = 75$ in favor of $H_a : \mu < 75$ if

$$z = \frac{\bar{x} - 75}{9/\sqrt{25}} < -z_{\alpha} = -z_{.01} = -2.327$$

or equivalently if

$$\bar{x} < 75 - 2.327(9/\sqrt{25}) = 70.8114$$

At level .01 with $\bar{x} = 72$ our test statistic gives z = -3/(9/5) = -5/3 = -1.667 which is not less than -2.327 so we do not reject the null hypothesis.

c) (5 points) Determine the probability of deciding $\mu = 75$ when in fact $\mu = 69$. I. e. compute the type II error probability $\beta(69)$.

Note that the acceptance region is determined by the null hypothesis "*Z*" = $\frac{\bar{X} - 75}{9/\sqrt{25}}$ statistic (whether or not it really holds; since here the null hypothesis fails this will be normal but not standard normal in which case we need to subtract the actual mean 69 from the sample mean to get a standard normal) .

$$\beta(69) = P(\text{ accept } H_0 \text{ when } \mu = 69)$$

$$P(-2.327 < \frac{\bar{X} - 75}{9/\sqrt{25}} = \frac{\bar{X} - 69}{9/\sqrt{25}} + \frac{69 - 75}{9/\sqrt{25}} = Z + \frac{69 - 75}{9/\sqrt{25}})$$

$$= P(Z > 3.33\bar{3} - 2.327 = 1.0063\bar{3})$$

$$= P(Z < -1.0063\bar{3}) = .1572 \qquad \text{(interpolating)}$$

d) (5 points) If the sample of *n* = 25 observations has a sample mean $\bar{x}=69$, determine the associated p-value.

The p value is the probability of seeing, under the assumption of the null hypothesis $H_0:\mu=75$, evidence as strong or stronger in favor of the alternative hypothesis as the given data (in this case the sample mean $\bar{x}=69$) provides. That is

$$p = P\left(Z\le\frac{69-75}{9/\sqrt{25}}=-3.33\bar{3}\right) = .0004$$

3. Tensile strength tests were carried out on two different grades of wire rod resulting in the data :

	sample size	sample mean (kg/mm^2)	sample S.D.
AISI 1064	$n_1=100$	$\bar{x}_1=106.4$	$s_1=1.2$
AISI 1078	$n_2=100$	$\bar{x}_2=124.4$	$s_1=2.2$

We would like to determine whether the actual population average strength μ_2 for the 1078 grade exceeds μ_1 (i.e. that for the 1064 grade) by more than 17.5 kg/mm^2

a) (5 points) In terms of the difference $\mu_1-\mu_2$ state the null and alternative hypotheses here.

We test $H_0:\mu_1-\mu_2=-17.5$ versus $H_a:\mu_1-\mu_2<-17.5$, the claim we want to establish.

b) (5 points) Give the large sample *Z* statistic for testing the difference of two population means (based on the approximately normal difference of sample means).

We standardize $\bar{X}_1-\bar{X}_2$ to get the large sample approximation $Z=\dfrac{\bar{X}_1-\bar{X}_2-(\mu_1-\mu_2)}{\sqrt{\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}}}$.

c) (5 points) Compute a 90% confidence interval for the difference $\mu_1-\mu_2$ of population means. For the Z statistic found above, we solve for $\mu_1-\mu_2$ the inequality (true with probability $1-\alpha$ = 90% prior to plugging in the particular sample)

$$-z_{.05}=-1.645<Z<z_{.05}=1.645$$

yielding $\bar{X}_1-\bar{X}_2-z_{.05}\sqrt{\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}} < \mu_1-\mu_2 < \bar{X}_1-\bar{X}_2-z_{.05}\sqrt{\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}}$

Plugging in our sample gives the 90% C. I.

$$106.4-124.4\pm1.645\sqrt{\frac{(1.2)^2+(2.2)^2}{100}} = -18\pm.1645\sqrt{(1.2)^2+(2.2)^2}$$
$$= -18\pm.41223 \quad = [-18.4122,-17.5878]$$

d) (5 points) Does the data provide compelling evidence for concluding that the average strength for the 1078 grade exceeds that for the 1064 grade by more than 17.5 kg/mm^2 ? I.e. should H_0 be rejected using significance level $\alpha=.05$? Explain.

We have a one sided test with alternative hypothesis value less than the null hypothesis value (to the left) so we reject H_0 if $Z<-z_\alpha=-z_{.05}=-1.645$ is sufficiently far to the left. Using the null hypothesis value of $\mu_1-\mu_2$ = - 17.5,

$$Z=\frac{106.4-124.4-(-17.5)}{\sqrt{\dfrac{(1.2)^2+(2.2)^2}{100}}} = \frac{-5}{\sqrt{(1.2)^2+(2.2)^2}} = -1.9952 < -z_\alpha=-1.645$$

Since the observed value is less than the z -critical value we reject H_0 and conclude there is sufficient evidence at significance level .05 that $H_a:\mu_1-\mu_2<-17.5$ holds.

4. If the speed of X= 75 out of 200 drivers exceed the 55 mph speed limit on a certain stretch of highway,

a) (5 points) What is your estimate $\hat{p}$ of the true proportion p of drivers whose speed exceeds the 55 mph speed limit ? What kind of random variable is X, the number of drivers whose speed exceeds 55 mph in the sample of size n = 200 and what are its mean and standard deviation?

Our estimate is $\hat{p} = X/n = 75/200 = 3/8 = .375$ here. X is a binomial random variable with mean np and standard deviation $\sqrt{np(1-p)}$

b) (5 points) Give an approximate 95% large sample confidence interval for the true proportion p of drivers who exceed the 55 mph limit on this stretch of highway. Explain what approximation allows you to say that $Z = \dfrac{X - np}{\sqrt{np(1-p)}} = \dfrac{(X/n) - p}{\sqrt{\dfrac{p(1-p)}{n}}}$ is approximately standard normal. The same should be approximately true if we estimate p by $\hat{p}$ in the denominator where $\hat{p}$ was obtained in part a)

Using the large sample normal approximation to the binomial random variable X, solving for the true population proportion p from the inequality $-z_{\alpha/2} < Z < z_{\alpha/2}$ with the above Z and using the estimate $\hat{p}$ to estimate the unknown p under the square root gives the approximate 95% interval $\dfrac{X}{n} \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ for p or plugging in the actual sample with $\hat{p} = \dfrac{75}{200} = \dfrac{3}{8} = .375$ gives $\dfrac{3}{8} \pm 1.96\sqrt{\dfrac{3}{8}(1-\dfrac{3}{8})/200} = .375 \pm .06709$

$$= [.3079, .4421]$$

c)(5 points) What's the estimated maximum error bound (with confidence $1-\alpha$ = 95%) on the difference $|\hat{p} - p|$ for the above sample ? (Hint : $-z_{\alpha/2} < Z < z_{\alpha/2}$ with Z as above).

Subtracting $\hat{p}$ from the above C. I. inequality for p and taking absolute values we find we are 95% confident that the maximum error is

$$|\hat{p} - p| < E = 1.96\sqrt{\frac{(3/8)(5/8)}{200}} = .06709$$

d) (5 points) How large a sample would we need to insure this maximum error is (with confidence $1-\alpha$ = 95%) less than 4% = .04 ? Use $\hat{p}$ to estimate p where needed.

Re-working the inequality

$$E = 1.96\frac{\sqrt{(3/8)(5/8)}}{\sqrt{n}} \le .04 \quad \text{gives} \quad n \ge (3/8)(5/8)\left(\frac{1.96}{.04}\right)^2 = 562.73 \text{ or } n \ge 563$$

5. Test are made on the proportion of defective castings produced by 2 different molds. If there were 12 defective castings among 100 made with Mold I and 45 defective castings made among 200 made with Mold II

Mold	I	II	totals
number of defectives :	12	45	57
number of non-defectives	88	155	243
sample sizes	100	200	300

a) (5 points) Give the pooled estimate $\hat{p}$ of the true proportion of defectives under the null hypothesis that the proportions are equal for both molds.

Under H_0 , the pooled estimate is $\hat{p} = \frac{57}{300} = \frac{19}{100} = .19$

b) (5 points) Using your pooled estimate found above, give the estimated expected numbers for the numbers of defectives and non-defectives for both molds.

The estimated expected numbers are

$100\,\hat{p}=19,\ \ 200\,\hat{p}=38,100(1-\hat{p})=100-19=81,\ \ 200(1-\hat{p})=200-38=162$

c) (5 points) Use the chi-squared statistic at significance level .01 to whether the true proportion of defectives is the same for both molds.

This is a one sided test. We reject the null hypothesis that the true proportion is the same for both molds if (with degrees of freedom *k*-1 = 1 here)

Reject H_0 if $\chi^2>\chi^2_{.01,1}=6.637$

$$\chi^2=\sum_{i,j}\frac{(o_{ij}-e_{ij})^2}{e_{ij}} = \frac{(12-19)^2}{19}+\frac{(45-38)^2}{38}+\frac{(88-81)^2}{81}+\frac{(155-162)^2}{162} = 4.7758<\chi^2_{.01,1}=6.637$$

hence we do not reject H_0 .

d) (5 points) What Z statistic would you use if you were using the Z test for a difference between the two proportions ? Hint : under the null hypothesis that there is no difference, the numerator which is the difference between the two sample proportions has expected value zero. For large samples as is the case here, the numerator is a linear combination of (approx) normals so is normal (approximately). Standardize this by first finding the variance of the numerator. Use the pooled estimator for the actual proportion.

$$Z=\frac{\hat{p}_1-\hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{\frac{12}{100}-\frac{45}{200}}{\sqrt{\frac{19}{100}\frac{81}{100}\left(\frac{1}{100}+\frac{1}{200}\right)}} = -2.185366894$$

Since this value lies between $-z_{.005}=-2.575$ and $z_{.005}=2.575$ we do not reject H_0 at level .01. Note also that the square of this *Z* value gives the chi-squared value found earlier : $(-2.185366894)^2=4.775828$. The *Z* test is two sided whereas the chi-squared test is one sided since a large *Z* value whether large and negative or large and positive when squared yields a chi-squared value which is large and positive