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Weak Convergence of Blockwise Bootstrapped Empirical Processes for  
Stationary Random Fields with Statistical Applications<sup>1 2</sup>

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<sup>2</sup>Proposed running head: Bootstrapped empirical processes for random fields.

In this article, we consider a stationary  $\alpha$ -mixing random field in  $\mathbb{R}^d$ . Under a large-sample scheme that is a mixture of the so-called “infill” and “increasing domain” asymptotics, we establish a functional central limit theorem for the empirical processes of this random field. Further, we apply a blockwise bootstrap to the samples. Under the condition that the side length of the block  $\lambda_l = \mathcal{O}(\lambda_n^\beta)$  for some  $0 < \beta < 1$ , where  $\lambda_n$  is the growth rate in the increasing domain asymptotics, we show that the bootstrapped empirical process converges weakly to the same limiting Gaussian process almost surely. Extension to multivariate random fields and application to differentiable statistical functionals are also given.

*Keywords and phrases:* Functional central limit theorem, increasing domain asymptotics, infill asymptotics, resampling,  $\alpha$ -mixing random field.

## 1 Introduction

Measurements over regions of interest are often modeled as a spatially distributed random process. In this article, we consider a random field (r.f.) with continuous spatial index,

$$\{Z(\mathbf{s}) : \mathbf{s} \in R\}, \quad (1.1)$$

where  $R \subset \mathbb{R}^d$  is a spatial region of interest and  $Z(\mathbf{s})$  is a random variable at each spatial location  $\mathbf{s} \in R$ . Associated with the r.f.  $\{Z(\cdot)\}$  at a given spatial location  $\mathbf{s}$ , the marginal cumulative distribution function (CDF) is defined as,

$$G(z; \mathbf{s}) \equiv \mathbb{P}(Z(\mathbf{s}) \leq z); \quad z \in \mathbb{R}. \quad (1.2)$$

Based on a finite sample  $\{Z(\mathbf{s}_1), \dots, Z(\mathbf{s}_N)\}$ , observed at known spatial sampling sites  $\{\mathbf{s}_1, \dots, \mathbf{s}_N\}$ , a commonly used estimator function of  $G(\cdot)$ , known as the empirical distribution function (EDF), is defined as,

$$F_N(z; R) \equiv N^{-1} \sum_{i=1}^N I(Z(\mathbf{s}_i) \leq z); \quad z \in \mathbb{R}. \quad (1.3)$$

The EDF is important, because a large class of statistics are its functionals. We shall show that the empirical process,

$$\xi_N(z; R) \equiv b_N [F_N(z; R) - G(z)]; \quad z \in \mathbb{R}, \quad (1.4)$$

converges weakly to a Gaussian process, under a suitable spatial asymptotic framework and a normalizing constant  $b_N$  to be determined later. Further, the sampling distribution of the limiting Gaussian process oftentimes involves unknown population quantities, such as the marginal CDF  $G(\cdot)$ , which need to be estimated separately. Here, we shall use an extension of a resampling method, known as the blockwise bootstrap, to spatial data and establish almost-sure weak convergence of the bootstrapped empirical process to the same limiting Gaussian process. Hence, large sample inference for the marginal CDF may be based on the blockwise bootstrap.

For i.i.d random variables, Bickel and Freedman (1981) establish almost-sure weak convergence of the bootstrapped empirical process for Efron’s (1979) resampling scheme. For stationary time series with short-range dependence, Bühlmann (1994) proves validity of the blockwise bootstrap of Künsch (1989) for the empirical processes of multivariate time series. For spatial data, after Hall (1985)’s first proposal of block resampling, Politis and Romano (1993) address block resampling schemes for general statistics. However, few results have been established for bootstrapped empirical processes for spatial data (Davison and Hinkley, 1996).

In this article, we consider a variant of the blockwise bootstrap method for spatial data due to Bühlmann and Künsch (1999) and show that the empirical process and the bootstrapped empirical

process have the same limiting distribution. To do so, we adopt some fairly general assumptions and a flexible asymptotic framework. We assume that the r.f. is stationary and its short-range dependence is specified via an  $\alpha$ -mixing condition. The spatial asymptotic structure used for the results is somewhat nonstandard. It is a combination of what are known as the “increasing domain asymptotics” and the “infill asymptotics” (cf. Cressie, 1993) and is similar to the asymptotic structures used by Lahiri (1999) for spatial CDF and by Hall and Patil (1994) for nonparametric estimation of the covariance function of a spatial process. Specializing the arguments to the “pure increasing domain” case, we also establish weak convergence of the empirical processes for a spatial process obtained on the integer grid  $\mathbb{Z}^d$  in  $\mathbb{R}^d$ . For proving the weak convergence results under the nonstandard spatial asymptotic framework, we do need to contend with some technical difficulties with boundary blocks of different shapes and sizes, which is not the case in time series. We describe the r.f. assumptions, spatial sampling and asymptotic framework, and the blockwise bootstrap in detail in Section 2.

The remaining paper is organized as follows. In Section 3, we state the weak convergence of the empirical process and the almost-sure weak convergence of the bootstrapped empirical process. Extension to multivariate r.f.s and application to differentiable statistical functionals are also given. In Section 4, we present the proofs.

## 2 Framework and Assumptions

In this section, we describe the assumptions on the r.f., the asymptotic framework, and the blockwise bootstrap method.

### 2.1 Random Field

We assume the following condition for the r.f. (1.1):

- (A.1) The r.f.  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$  is stationary. Hence, the marginal CDF is spatial location invariant and is denoted by

$$G(z) \equiv G(z; \mathbf{0}); \quad z \in \mathbb{R}, \quad (2.1)$$

where  $G(z; \mathbf{0})$  is defined in (1.2) and  $\mathbf{0}$  is the origin in  $\mathbb{R}^d$ . Henceforth, we assume that  $G(\cdot)$  is continuous.

Note that assumption (A.1) assures that the marginal CDF is smooth, but we do not require knowledge of its parametric forms. By stationarity, the joint bivariate distribution of  $Z(\cdot)$  is spatial lag-shift invariant and is denoted by,

$$H(z_1, z_2; \mathbf{s}) \equiv \mathbb{P}(Z(\mathbf{0}) \leq z_1, Z(\mathbf{s}) \leq z_2); \quad z_1, z_2 \in \mathbb{R}, \quad \mathbf{s} \in \mathbb{R}^d. \quad (2.2)$$

Further, we use an  $\alpha$ -mixing coefficient to specify the dependence structure of the r.f.  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$  (cf. Doukhan, 1994, pp.3 and pp.15). For  $S_1, S_2 \subset \mathbb{R}^d$ , let

$$\alpha_1(S_1, S_2) \equiv \sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_Z(S_1), B \in \mathcal{F}_Z(S_2) \right\},$$

where  $\mathcal{F}_Z(S)$  is the  $\sigma$ -algebra generated by the variables  $\{Z(\mathbf{s}) : \mathbf{s} \in S\}$ ;  $S \subset \mathbb{R}^d$ . Then the  $\alpha$ -mixing coefficient of the r.f.  $Z(\cdot)$  is defined as,

$$\alpha(k; b) \equiv \sup \left\{ \alpha_1(S_1, S_2) : S_1, S_2 \in \mathfrak{R}_p(b), \delta(S_1, S_2) \geq k \right\}. \quad (2.3)$$

Here the class of sets  $\mathfrak{R}_p(b) \equiv \{\cup_{i=1}^p D_i : \sum_{i=1}^p |D_i| \leq b\}$  denotes the collection of all disjoint union of  $p$  cubes  $D_1, \dots, D_p$  in  $\mathbb{R}^d$ ,  $p$  is a positive integer, and  $b > 0$ . As it turns out, we shall use  $p = 6r_0$ ,

as it will be suitable for computing the  $(6r_0)$ -th moment bounds by Theorem 1.4.1.1 of Doukhan (1994), where  $r_0$  is a positive integer to be specified later. Further,  $\delta(S_1, S_2) \equiv \inf\{\|\mathbf{x} - \mathbf{s}\| : \mathbf{x} \in S_1, \mathbf{s} \in S_2\}$  denotes the distance between the sets  $S_1, S_2 \subset \mathbb{R}^d$ , where  $\|\cdot\|$  is the  $\ell^1$ -norm on  $\mathbb{R}^d$  defined by  $\|\mathbf{x}\| = \sum_{i=1}^d |x_i|$ ;  $\mathbf{x} \equiv (x_1, \dots, x_d)' \in \mathbb{R}^d$ . We shall further suppose that

(A.2) There exist positive real numbers  $\tau, \theta$ , such that  $\alpha(k; b) \leq \text{const.} k^{-\tau} b^\theta$ .

A similar strong mixing condition has been used by Doukhan (1994) (cf. Chapter 2) and Lahiri (2000) to obtain moment bounds on the weighted sums of r.f. variables  $Z(\cdot)$ . In the following, we shall use assumption (A.2) under different conditions on  $\tau$  and  $\theta$  which will be specified in the statement of each result separately.

## 2.2 Asymptotic Framework

Several different asymptotic structures have been used to study large-sample properties of estimators and predictors based on spatial data. Oftentimes, they come from different combinations of two basic sampling forms. One form is known as ‘‘increasing domain asymptotics’’, where all sampling sites are separated by a fixed positive distance and the sampling region becomes unbounded as the sample size increases. The other form is called ‘‘infill asymptotics’’, where more and more samples are collected within a bounded sampling region (cf. Cressie, 1993). For our purpose, we use a mixture of these two forms for the asymptotic structure, where we allow the sampling region to grow and, simultaneously, allow infilling of any fixed bounded subregion. Other papers, where a similar asymptotic structure has been used, are Härdle and Tuan (1986), Hall and Patil (1994), and Lahiri (1999).

For the increasing domain component of the asymptotic structure, let  $R_0$  denote an open connected subset of  $(-1/2, 1/2]^d$  that contains the origin  $\mathbf{0}$ . For any sequence of positive real numbers  $\{a_n\}$  tending to 0 as  $n \rightarrow \infty$ , let  $R_0$  satisfy the condition that:

(A.3) the number of cubes of the integer lattice  $a_n \mathbb{Z}^d$  that intersect both  $R_0$  and its complement  $R_0^c$  is of the order  $\mathcal{O}(a_n^{-(d-1)})$ , as  $n \rightarrow \infty$ .

Then we obtain the sampling region  $R_n$  by inflating the region  $R_0$  with  $R_n \equiv \lambda_n R_0$ , where  $\{\lambda_n\}$  denotes a scaling sequence of positive real numbers tending to infinity as  $n \rightarrow \infty$ . Because  $R_0$  contains the origin  $\mathbf{0}$ , the shape of the sampling region is preserved for different values of  $n$ . Moreover, the number-of-cubes requirement of  $R_0$  in (A.3) guarantees that the effect of the sampling sites on the boundary of  $R_n$  is negligible compared to the totality of all sampling sites in  $R_n$ .

For the infill component of the asymptotic structure, we partition the sampling region  $R_n$  into equal-volume cubes and denote them by  $\Gamma(\mathbf{i}) \equiv (\mathbf{i} + (0, 1]^d)h_n$ ;  $\mathbf{i} \in \mathbb{Z}^d$ , where  $\{h_n\}$  denotes a sequence of positive real numbers decreasing to 0 as  $n \rightarrow \infty$ , and  $\Gamma(\mathbf{i}) \cap R_n \neq \emptyset$ . By allowing exactly one potential sampling site in each cube  $\Gamma(\mathbf{i})$ , we can form a regular square grid of sampling sites. Specifically, let  $\mathbf{c}$  denote an arbitrary point in the interior of the unit cube  $(0, 1]^d$  and assign  $\mathbf{c}h_n$  in the cube  $(0, h_n]^d$  to be the starting sampling site. Then, the sampling sites are given by the translated points,  $(\mathbf{i} + \mathbf{c})h_n \in \Gamma(\mathbf{i})$ ;  $\mathbf{i} \in \mathbb{Z}^d$ , that lie in the sampling region  $R_n$ . That is, the sampling sites in  $R_n$  form a grid,  $\mathcal{S}_n \equiv \{(\mathbf{i} + \mathbf{c})h_n \in \Gamma(\mathbf{i}) \cap R_n, \mathbf{i} \in \mathbb{Z}^d\}$ ; and as  $n \rightarrow \infty$ , the number of sampling sites  $N \equiv N_n$  in  $R_n$  satisfies the growth condition,  $N_n \sim |R_0| \lambda_n^d h_n^{-d}$ . Here, for any two sequences of positive real numbers  $\{a_n\}$  and  $\{b_n\}$ , we write  $a_n \sim b_n$  to denote  $a_n/b_n \rightarrow 1$ , as  $n \rightarrow \infty$ .

For notational convenience, we let  $F_n(\cdot) \equiv F_{N_n}(\cdot; R_n)$  and  $\xi_n(\cdot) \equiv \xi_{N_n}(\cdot; R_n)$ . As it turns out, the normalizing constant for  $\xi_n$  is  $b_n \equiv b_{N_n} = \lambda_n^{d/2}$ . Hence,

$$F_n(z) = N_n^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} I(Z(\mathbf{s}) \leq z); \quad z \in \mathbb{R}, \quad (2.4)$$

$$\xi_n(z) = \lambda_n^{d/2} [F_n(z) - G(z)]; \quad z \in \mathbb{R}. \quad (2.5)$$

In summary, we assume that,

$$(A.4) \quad \lambda_n \rightarrow \infty \text{ and } h_n \rightarrow 0, \text{ as } n \rightarrow \infty.$$

### 2.3 Blockwise Bootstrap

In this section, we consider a blockwise bootstrap method under the asymptotic framework in Section 2.2. This bootstrap method is a variant of the method suggested by Bühlmann and Künsch (1999) (also, cf. Künsch, 1989; Liu and Singh, 1992; Politis and Romano, 1993; Bühlmann, 1994) and can be used to estimate the sampling distribution of the EDF  $F_n(\cdot)$  and the empirical process  $\xi_n(\cdot)$ , defined in (2.4) and (2.5).

The basic idea is to produce replicates of the r.f.  $Z(\cdot)$  over the region  $R_n$ . First, we partition the region  $R_n$  into blocks of the form  $R_{n,\mathbf{k}} \equiv (\mathbf{k} + [0, 1)^d)\lambda_l \cap R_n$ ;  $\mathbf{k} \in \mathbb{Z}^d$ . Here  $\{\lambda_l\}$  denotes a scaling sequence of positive real numbers tending to infinity as  $n \rightarrow \infty$ . Hence  $\lambda_l$  controls the size of the blocks. Let  $\mathcal{K}_n \equiv \{\mathbf{k} \in \mathbb{Z}^d : (\mathbf{k} + [0, 1)^d)\lambda_l \cap R_n \neq \emptyset\}$  denote the index set of square blocks that have sides of length  $\lambda_l$  and that intersect  $R_n$ . Then, we have,  $R_n = \cup_{\mathbf{k} \in \mathcal{K}_n} R_{n,\mathbf{k}}$ . Bühlmann and Künsch (1999) propose a more general partition of  $R_n$  where disjoint blocks are of similar shape and size, in order to handle  $R_n$  of general and irregular shape. As it turns out, partitioning into squares  $R_{n,\mathbf{k}}$  would work for  $R_n$  that is of general shape, provided that the boundary of  $R_n$  satisfies the condition (A.3).

Now, we define the index set for all possible translations of  $R_{n,\mathbf{0}}$  at an increment  $h_n$  as,

$$\mathcal{T}_n \equiv \{\mathbf{t} \in \mathbb{Z}^d : \mathbf{t}h_n + R_{n,\mathbf{0}} \subset R_n\}.$$

Denote the collection of sampling sites in  $R_{n,\mathbf{k}}$  by  $\mathcal{B}_{n,\mathbf{k}} \equiv R_{n,\mathbf{k}} \cap \mathcal{S}_n$  and its translation back to the origin by  $\mathcal{B}_{n,\mathbf{k}}^{\mathbf{0}} \equiv \mathcal{B}_{n,\mathbf{k}} - \lambda_l \mathbf{k}$ . Then, for the square blocks  $R_{n,\mathbf{k}}$  completely contained in  $R_n$ , the set  $\{\mathbf{t}h_n + \mathcal{B}_{n,\mathbf{k}}^{\mathbf{0}} : \mathbf{t} \in \mathcal{T}_n\}$  consists of all possible translations of  $\mathcal{B}_{n,\mathbf{k}}$  within  $R_n$  at the increment  $h_n$ . For boundary blocks  $R_{n,\mathbf{k}}$ , for simplicity, we restrict attention to only the collection  $\{\mathbf{t}h_n + \mathcal{B}_{n,\mathbf{k}}^{\mathbf{0}} : \mathbf{t} \in \mathcal{T}_n\}$  out of all possible translates of  $\mathcal{B}_{n,\mathbf{k}}$  in  $R_n$ .

The main idea of the blockwise bootstrap is to select *randomly with replacement* a set  $\mathcal{B}_{n,\mathbf{k}}^*$  from  $\{\mathbf{t}h_n + \mathcal{B}_{n,\mathbf{k}}^{\mathbf{0}} : \mathbf{t} \in \mathcal{T}_n\}$ , for each  $\mathbf{k} \in \mathcal{K}_n$ . Hence, the resamples for the region  $R_{n,\mathbf{k}}$  are the data values lying in the randomly selected block  $\mathcal{B}_{n,\mathbf{k}}^*$ , that is,  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathcal{B}_{n,\mathbf{k}}^*\}$ . Since  $\{R_{n,\mathbf{k}} : \mathbf{k} \in \mathcal{K}_n\}$  is a partition of  $R_n$ ,  $\{Z(\mathbf{s}) : \mathbf{s} \in \cup_{\mathbf{k} \in \mathcal{K}_n} \mathcal{B}_{n,\mathbf{k}}^*\}$  gives a set of resamples of the observations  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathcal{S}_n\}$  over all of  $R_{n,\mathbf{k}}$  on the grid  $\mathcal{S}_n$ . We define the bootstrap version of the EDF (2.4) as,

$$F_n^*(z) \equiv \left( \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{n,\mathbf{k}}^*| \right)^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \sum_{\mathbf{s} \in \mathcal{B}_{n,\mathbf{k}}^*} I(Z(\mathbf{s}) \leq z); \quad z \in \mathbb{R}. \quad (2.6)$$

Hence, the bootstrap version of the empirical process (2.5) is denoted by,

$$\xi_n^*(z) \equiv \lambda_n^{d/2} [F_n^*(z) - E_*(F_n^*(z))]; \quad z \in \mathbb{R}, \quad (2.7)$$

where  $E_*(\cdot)$  denotes the expectation conditional on the sample  $\{Z(\mathbf{s}_1), \dots, Z(\mathbf{s}_{N_n})\}$ .

To establish the almost-sure weak convergence, we make the following assumption on the block scaling factor  $\lambda_l$  relative to the increasing domain rate  $\lambda_n$ .

$$(A.5) \quad \lambda_l \rightarrow \infty, \lambda_l = O(\lambda_n^\beta) \text{ as } n \rightarrow \infty, \text{ for some } 0 < \beta < 1, \text{ and } \sum_{n=1}^{\infty} \lambda_n^{-\zeta d} < \infty \text{ for any } \zeta > 1.$$

It is known that the optimal (in the mean square error sense) choice of  $\lambda_l$  for estimation of variance type functional is  $c\lambda_n^{d/(d+2)}$  for some constant  $c > 0$  (cf. Politis and Romano (1994)). Assumption (A.5) allows the bootstrap block scaling factor  $\lambda_l$  to grow at this optimal rate.

### 3 Main Results

In this section, we present the main theoretical results and their statistical applications. In Section 3.1, weak convergence of the empirical process is established. The blockwise bootstrap method is applied in Section 3.2 and weak convergence of the bootstrapped empirical process is shown. In Section 3.3, the results are extended to multivariate r.f.s and two applications are presented in Section 3.4.

#### 3.1 Weak Convergence of Empirical Processes

The first main result is a functional central limit theorem (CLT) for the empirical process  $\xi_n(\cdot)$  defined by (2.5). We consider  $\xi_n$  as a random element of the space  $D[-\infty, \infty]$  of real-valued functions on  $[-\infty, \infty]$  that are right continuous with left-hand limits. We equip  $D[-\infty, \infty]$  with the Skorohod  $J_1$ -topology and establish the weak convergence of  $\xi_n$ .

**Theorem 3.1** *Under the asymptotic framework of Section 2.2, suppose that assumptions (A.1)–(A.4) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > 2d$  and  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$\xi_n \xrightarrow{\mathcal{D}} \mathcal{W}.$$

Here  $\xrightarrow{\mathcal{D}}$  denotes weak convergence of  $D[-\infty, \infty]$ -valued random variables,  $\mathcal{W}$  is a zero-mean Gaussian process on  $[-\infty, \infty]$  with covariance function,

$$\text{Cov}(\mathcal{W}(z_1), \mathcal{W}(z_2)) = |R_0|^{-1} \int_{\mathbb{R}^d} [H(z_1, z_2; \mathbf{s}) - G(z_1)G(z_2)] d\mathbf{s}; \quad z_1, z_2 \in \mathbb{R}.$$

Moreover, the sample paths of  $\mathcal{W}$  are continuous with probability 1.

Theorem 3.1 shows that, under the given asymptotic structure on a stationary r.f., the empirical process  $\xi_n$  converges weakly to a Gaussian process that has zero mean and covariance function depending on the marginal CDF (2.1) and the bivariate distributions (2.2). Note also that the normalizing constant depends on the increasing domain rate  $\lambda_n$  and does not depend on the infill rate  $h_n$ .

As it turns out, Theorem 3.1 and the other results presented in Sections 3.2–3.4 continue to hold for the pure increasing domain asymptotics under identical regularity conditions of the respective theorems.

**Remark 3.1** *It follows from the proof of Theorem 3.1 that the weak convergence of  $\xi_n$  to  $\mathcal{W}$  still holds if  $h_n \equiv 1$  for all  $n \geq 1$ .*

When  $h_n \equiv 1$ , this case corresponds to the pure increasing domain asymptotics, under which weak convergence has been widely studied. In particular, when  $d = 1$ , the r.f.  $Z(\cdot)$  reduces to a sequence of random variables  $\{Z_1, Z_2, \dots\}$ . Deo (1973) establishes weak convergence of empirical processes for a strong-mixing sequence of random variables. Sen (1974) and Yoshihara (1975) show convergence results of multidimensional empirical processes for mixing sequences of random vectors. More recently, several authors present convergence of the empirical processes based on dependent random variables. For example, Arcones and Yu (1994) give convergence conditions for  $\beta$ -mixing sequences indexed by the V-C subgraph classes of functions.

For the more general case  $d \geq 1$ , Guyon (1995) presents CLT results for the sums of a strong-mixing r.f. on an integer lattice under the pure increasing domain asymptotics. Lahiri (2000) establishes CLT for weighted sums of such r.f.s under the mixture of the increasing domain asymptotics and the infill asymptotics, for both fixed and random sampling designs. To establish Theorem 3.1, we shall use similar arguments as in Lahiri (2000). Moment bounds and techniques presented in Doukhan (1994) for r.f.s and in Billingsley (1968) for random sequences will also be utilized.

### 3.2 Weak Convergence of Bootstrapped Empirical Processes

The second main result establishes the weak convergence of the bootstrapped empirical process. Let  $r_0 \equiv \min\{k \in \mathbb{Z} : k \geq 3 \text{ and } k(1 - \beta) \geq 3/2\}$ , where  $\beta$  is defined in the assumption (A.5). Also, define  $\tau_1^0 \equiv d(2r_0 - 1)(r_0 + 1)$ ,  $\tau_2^0 \equiv d(4r_0 + 3)[1 + 2(r_0 + 1)(4\beta - 1)_+/(7 - 6\beta)]$ , and  $\tau_3^0 \equiv 31d[1 + 48(2\beta - 1)_+/(7 - 6\beta)]$ , where for any  $x \in \mathbb{R}$ ,  $x_+ \equiv \max\{x, 0\}$ . Note that for  $0 < \beta \leq 1/4$ ,  $\tau_2^0 = d(4r_0 + 3)$  and for  $0 < \beta \leq 1/2$ ,  $\tau_3^0 = 31d$ .

**Theorem 3.2** *Suppose that the conditions in Theorem 3.1 hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \max\{77d, \tau_1^0, \tau_2^0, \tau_3^0\}$  and  $\theta < \tau/d$ . Furthermore, assume the bootstrap procedure given in Section 2.3, suppose assumption (A.5) holds. Then, there exists a set  $A \in \mathcal{F}$  with  $P(A) = 1$  such that,*

$$\xi_n^* \xrightarrow{\mathcal{D}} \mathcal{W},$$

as  $n \rightarrow \infty$  on  $A$ . Here  $\xrightarrow{\mathcal{D}}$  continues to denote weak convergence of  $D[-\infty, \infty]$ -valued random variables and  $\mathcal{W}$  is defined in Theorem 3.1.

Theorem 3.2 shows that the bootstrap version of the empirical process converges weakly to the same limiting Gaussian process as that in Theorem 3.1. Hence, in practice, one can use the bootstrapped resamples  $\{Z(\mathbf{s}) : \mathbf{s} \in \cup_{\mathbf{k} \in \mathcal{K}_n} \mathcal{B}_{n, \mathbf{k}}^*\}$  to estimate the sampling distribution of the empirical process  $\xi_n$  and the asymptotic correctness is guaranteed. For smaller block scaling factor  $\lambda_l$  with  $0 < \beta \leq 1/4$ , the almost-sure weak convergence of the bootstrapped empirical process holds in any dimension  $d \geq 1$ , if  $\tau > 77d$ . Note that the optimal subregion size in dimension  $d$  is given by  $\lambda_l \sim \text{const.} \lambda_n^{d/(d+2)}$  (cf. Politis and Romano, 1994). Thus, for  $d = 2$ , the optimal  $\lambda_l \sim \text{const.} \lambda_n^{1/2}$  and it can be checked that Theorem 3.2 remains valid for this choice of  $\lambda_l$  (with  $\beta = 1/2$ ), if  $\tau > 77d$ . For dimension  $d = 3$ , the optimal choice of  $\lambda_l$  corresponds to  $\beta = 3/5$  and this requires that  $\tau > 118.53d$ . The condition on  $\tau$  here is determined by repeated application of the spatial version of Rosenthal's inequality given by Doukhan (1994) (cf. Section 1.4).

For a sequence of stationary and dependent random vectors,  $\{\mathbf{Z}_1, \mathbf{Z}_2, \dots \in \mathbb{R}^m\}$ , where  $m \geq 1$ , under the condition that the block length  $l(n) = \mathcal{O}(n^{1/2-\epsilon})$  with  $\epsilon > 0$ , Bühlmann (1994) proves that the bootstrapped vector empirical process converges weakly almost surely to the same Gaussian process to which the original empirical process converges. Naik-Nimbalkar and Rajarshi (1994) give similar results for a sequence of random variables (i.e., for  $m = 1$ ) and use somewhat restrictive assumptions such that the weak convergence is achieved either in probability or almost surely, depending on the block length. Specifically, almost-sure weak convergence is obtained if  $l(n) = \mathcal{O}(n^{1/2-\epsilon})$  for  $1/4 < \epsilon < 1/2$ ; and only convergence in probability is attained if  $l(n) = \mathcal{O}(n^{1/2-\epsilon})$  for  $0 < \epsilon \leq 1/4$ . In comparison, our result is valid for all block lengths  $l(n) = \mathcal{O}(n^\beta)$  for any  $\beta \in (0, 1)$  in the one-dimensional case, and for all subregions with  $\lambda_l = \mathcal{O}(\lambda_n^\beta)$  for any  $\beta \in (0, 1)$  in higher dimensions.

In Section 4.2, we extend the arguments in Bühlmann (1994) and present the proof of Theorem 3.2 for the r.f. case. The two main steps are to establish almost-sure weak convergence of the finite dimensional distribution of  $\xi_n^*$  and the tightness of  $\xi_n^*$ , as in Lemmas 3 and 11 of Bühlmann (1994).

### 3.3 Extension to Multivariate Random Fields

For some applications, an extension of Theorem 3.1 and 3.2 to the multivariate case is important. In this section, we consider an  $m$ -dimensional ( $m \geq 1$ ) stationary r.f.  $\{\mathbf{Z}(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$  with components,

$$\mathbf{Z}(\mathbf{s}) = (Z_1(\mathbf{s}), \dots, Z_m(\mathbf{s}))', \tag{3.1}$$

and prove weak convergence of the EDF,

$$F_n^{(m)}(\mathbf{z}) \equiv N_n^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} I(\mathbf{Z}(\mathbf{s}) \leq \mathbf{z}); \quad \mathbf{z} \in \mathbb{R}^m, \quad (3.2)$$

and its block bootstrap version under the spatial asymptotic structures of Section 2, where for any two vectors  $\mathbf{u} = (u_1, \dots, u_m)'$  and  $\mathbf{v} = (v_1, \dots, v_m)'$ ,  $m \geq 1$ , we write  $\mathbf{u} \leq \mathbf{v}$  if  $u_i \leq v_i$  for all  $1 \leq i \leq m$ . As in the univariate case (2.6), the block bootstrap version of the EDF  $F_n^{(m)}$  is given by,

$$F_n^{(m)*}(\mathbf{z}) \equiv \left( \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{n,\mathbf{k}}^*| \right)^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \sum_{\mathbf{s} \in \mathcal{B}_{n,\mathbf{k}}^*} I(\mathbf{Z}(\mathbf{s}) \leq \mathbf{z}); \quad \mathbf{z} \in \mathbb{R}^m, \quad (3.3)$$

where  $\mathcal{B}_{n,\mathbf{k}}^*$ ;  $\mathbf{k} \in \mathcal{K}_n$  are defined as in Section 2.3. Define the normalized versions of  $F_n^{(m)}$  and  $F_n^{(m)*}$  as,

$$\xi_n^{(m)}(\mathbf{z}) \equiv \lambda_n^{d/2} [F_n^{(m)}(\mathbf{z}) - G^{(m)}(\mathbf{z})]; \quad \mathbf{z} \in \mathbb{R}^m, \quad (3.4)$$

$$\xi_n^{(m)*}(\mathbf{z}) \equiv \lambda_n^{d/2} [F_n^{(m)*}(\mathbf{z}) - E_*(F_n^{(m)*}(\mathbf{z}))]; \quad \mathbf{z} \in \mathbb{R}^m, \quad (3.5)$$

respectively, where,

$$G^{(m)}(\mathbf{z}) \equiv P(\mathbf{Z}(\mathbf{0}) \leq \mathbf{z}); \quad \mathbf{z} \in \mathbb{R}^m, \quad (3.6)$$

is the marginal distribution function of  $\mathbf{Z}(\cdot)$  under stationarity. Then,  $\xi_n^{(m)}$  and  $\xi_n^{(m)*}$  belong to the space  $D[-\infty, \infty]^m$  of real-valued functions on  $[-\infty, \infty]^m$  that are continuous from above with limits from below. We equip  $D[-\infty, \infty]^m$  with the extended Skorohod  $J_1$ -topology (Bickel and Wichura, 1971) and establish weak convergence of  $\xi_n^{(m)}$  and  $\xi_n^{(m)*}$  to a Gaussian process. Now, let  $\xrightarrow{\mathcal{D}}$  denote weak convergence of  $D[-\infty, \infty]^m$ -valued random variables. Also, let  $\alpha^{(m)}(\cdot, \cdot)$  denote the strong-mixing coefficient of  $\mathbf{Z}(\cdot)$  defined by (2.3) with  $\mathcal{F}_Z(S)$  replaced by  $\mathcal{F}_{\mathbf{Z}}(S)$ ;  $S \subset \mathbb{R}^d$ .

**Theorem 3.3** *Let  $\{\mathbf{Z}(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$  denote a stationary  $m$ -dimensional multivariate r.f., let the  $m$ -components of  $\mathbf{Z}(\mathbf{0})$  have continuous marginal distributions on  $\mathbb{R}$ , assume  $\alpha$ -mixing assumption (A.2), and the parameters  $\tau$  and  $\theta$  of the mixing coefficient  $\alpha^{(m)}(\cdot, \cdot)$  satisfy  $\tau \geq 3d(d+2)m + 8d$  and  $\theta < \tau/d$ .*

- (i) *Under the asymptotic framework of Section 2.2, further suppose that assumptions (A.3) and (A.4) hold. Then, as  $n \rightarrow \infty$ ,*

$$\xi_n^{(m)} \xrightarrow{\mathcal{D}} \mathcal{W}^{(m)}, \quad (3.7)$$

where  $\mathcal{W}^{(m)}$  is a zero-mean Gaussian process on  $[-\infty, \infty]^m$  with covariance function,

$$\text{Cov}(\mathcal{W}^{(m)}(\mathbf{z}_1), \mathcal{W}^{(m)}(\mathbf{z}_2)) = |R_0|^{-1} \int_{\mathbb{R}^d} [H^{(m)}(\mathbf{z}_1, \mathbf{z}_2; \mathbf{s}) - G^{(m)}(\mathbf{z}_1)G^{(m)}(\mathbf{z}_2)] d\mathbf{s},$$

and bivariate distribution,

$$H^{(m)}(\mathbf{z}_1, \mathbf{z}_2; \mathbf{s}) \equiv P(\mathbf{Z}(\mathbf{0}) \leq \mathbf{z}_1, \mathbf{Z}(\mathbf{s}) \leq \mathbf{z}_2); \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^m, \mathbf{s} \in \mathbb{R}^d.$$

Further, the sample paths of  $\mathcal{W}^{(m)}$  are continuous with probability 1.

- (ii) *Moreover, assume the bootstrap procedure given in Section 2.3, suppose assumption (A.5) holds. Then, there exists a set  $A \in \mathcal{F}$  with  $P(A) = 1$  such that,*

$$\xi_n^{(m)*} \xrightarrow{\mathcal{D}} \mathcal{W}^{(m)}, \quad (3.8)$$

as  $n \rightarrow \infty$  on  $A$ .

### 3.4 Applications

Results of Sections 3.1–3.3 readily allow one to establish validity of the spatial block bootstrap for estimators that are smooth functionals of the empirical process of a multivariate r.f.  $\{\mathbf{Z}(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ . Suppose that,

$$\boldsymbol{\theta} \equiv \mathbf{T}(G^{(m)}), \quad (3.9)$$

is a parameter of interest where  $\mathbf{T}(\cdot)$  is an  $\mathbb{R}^p$ -valued ( $p \geq 1$ ) functional defined on a rich subset  $\mathcal{G}_m$  of the set  $\mathcal{P}_m$  of all probability measures in  $\mathbb{R}^m$ . Then, a natural estimator of  $\boldsymbol{\theta}$  is given by,

$$\hat{\boldsymbol{\theta}}_n = \mathbf{T}(F_n^{(m)}), \quad (3.10)$$

where  $F_n^{(m)}$  is the EDF of  $\{\mathbf{Z}(\mathbf{s}) : \mathbf{s} \in \mathcal{S}_n\}$ . A few important examples of such estimators are listed below.

#### Example 3.1 *The Sample Mean*

Here  $p = 1$ ,  $m = 1$ ,  $T(G) = \int zdG(z)$ , and  $G \in \mathcal{G} \equiv \{F \in \mathcal{P} : \int |x|dF(x) < \infty\}$ . In this case,  $\theta = E(Z(\mathbf{0}))$  and  $\hat{\theta}_n = N_n^{-1} \sum_{i=1}^{N_n} Z(\mathbf{s}_i)$ .

#### Example 3.2 *M-estimators*

Let  $\Psi : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^p$  be a function and let,

$$\begin{aligned} \mathcal{G}_\Psi &\equiv \{F^{(m)} \in \mathcal{P}_m : \int |\Psi(\mathbf{x}, \mathbf{t})|dF^{(m)}(\mathbf{x}) < \infty \text{ for all } \mathbf{t} \in \mathbb{R}^p \\ &\text{and the equation } \int \Psi(\mathbf{x}; \mathbf{t})dF^{(m)}(\mathbf{x}) = \mathbf{0} \text{ for a unique solution}\}. \end{aligned}$$

Let the parameter of interest  $\boldsymbol{\theta}$  be defined by (3.9) with the functional  $\mathbf{T}(\cdot)$  given by,

$$\int \Psi(\mathbf{x}; \mathbf{T}(G^{(m)}))dG^{(m)}(\mathbf{x}) = \mathbf{0}. \quad (3.11)$$

Then, the estimator  $\hat{\boldsymbol{\theta}}_n$  of  $\boldsymbol{\theta}$  is a solution to the equation,

$$\sum_{\mathbf{s} \in \mathcal{S}_n} \Psi(\mathbf{Z}(\mathbf{s}); \hat{\boldsymbol{\theta}}_n) = \mathbf{0}. \quad (3.12)$$

Under suitable regularity conditions, maximum likelihood estimators of parameters in parametric spatial models (e.g., parametric Markov r.f. models) may be expressed in the above form. Also, estimators defined through estimating equations in spatial semiparametric models (e.g., spatial autoregression models and certain geostatistical processes with parametric variogram models) are also covered by this example. In such applications, given a univariate r.f.  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ , it may be appropriate to consider the multivariate r.f.  $\mathbf{Z}(\mathbf{s}) \equiv (Z(\mathbf{s}), Z(\mathbf{s} + \mathbf{s}_2), \dots, Z(\mathbf{s} + \mathbf{s}_m))'$ ,  $\mathbf{s} \in \mathbb{R}^d$  for a given set of lag vectors  $\mathbf{s}_2, \dots, \mathbf{s}_m \in \mathbb{R}^d$ .

For establishing validity of the block bootstrap method, we shall assume that the functional is Fréchet differentiable at  $G^{(m)}$  under the Kolmogorov distance in the sense that there exists a Borel measurable function  $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^p$  such that,

$$\lim_{n \rightarrow \infty} \|\mathbf{T}(G_n^{(m)}) - \mathbf{T}(G^{(m)}) - \int \mathbf{g}d(G_n^{(m)} - G^{(m)})\| / \|G_n^{(m)} - G^{(m)}\|_\infty = 0, \quad (3.13)$$

for any sequence  $\{G_n^{(m)}\}$  of probability measures in  $\mathcal{P}_m$  converging to  $G^{(m)}$  in the Kolmogorov distance  $\|\cdot\|_\infty$ , defined by,

$$\|G_n^{(m)} - G^{(m)}\|_\infty = \sup\{G_n^{(m)}(-\infty\mathbf{1}, \mathbf{x}] - G^{(m)}(-\infty\mathbf{1}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}.$$

For the sample mean of Example 3.1, it is easy to see that the functional  $T(\cdot)$  is Fréchet differentiable at  $G$  and the function  $g$  is given by  $g(x) = x$ ;  $x \in \mathbb{R}$ . Under some regularity conditions on the function  $\Psi$ , Fréchet differentiability of the functional  $\mathbf{T}(\cdot)$  under Kolmogorov distance in Example 3.2 (defining the M-estimators  $\hat{\theta}_n$  of (3.10)) is known to be true (see Clarke, 1983; Fernholz, 1983; and Bühlmann, 1994). The following result shows that the spatial block bootstrap provides a valid approximation to the distribution of normalized  $\hat{\theta}_n$ . Let  $\theta_n^* \equiv \mathbf{T}(F_n^{(m)*})$  denote the block bootstrap version of  $\hat{\theta}_n$  and let  $\tilde{\theta}_n \equiv \mathbf{T}(E_*(F_n^{(m)*}))$ , where  $F_n^{(m)*}(\cdot)$  is as defined in (3.3).

**Theorem 3.4** *Suppose that the conditions of Theorem 3.3 hold, and that the functional  $\mathbf{T}$  defining  $\theta$  and  $\hat{\theta}$  of (3.9) and (3.10) is Fréchet differentiable at  $G^{(m)}$ . Further, suppose that the function  $g$  of (3.13) is such that  $E\|g(\mathbf{Z}(\mathbf{0}))\|^{12} < \infty$ . Then, for  $\theta$  and  $\hat{\theta}_n$ , as  $n \rightarrow \infty$ ,*

(i)

$$\lambda_n^{d/2}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \Sigma_0), \quad (3.14)$$

where  $\Sigma_0 \equiv |R_0|^{-1} \int_{\mathbb{R}^d} E[g(\mathbf{Z}(s))g(\mathbf{Z}(\mathbf{0}))'] ds$ ;

(ii)

$$\sup_{C \in \mathcal{C}} \left| P_*(\lambda_n^{d/2}(\theta_n^* - \tilde{\theta}_n) \in C) - P(\lambda_n^{d/2}(\hat{\theta}_n - \theta) \in C) \right| = o_p(1), \quad (3.15)$$

for any collection  $\mathcal{C}$  of measurable convex sets in  $\mathbb{R}^m$ .

Thus, it follows from Theorem 3.4 that,

$$\sup_{\mathbf{x} \in \mathbb{R}^p} \left| P_*(\lambda_n^{d/2}(\theta_n^* - \tilde{\theta}_n) \leq \mathbf{x}) - P(\lambda_n^{d/2}(\hat{\theta}_n - \theta) \leq \mathbf{x}) \right| = o_p(1). \quad (3.16)$$

The convergence of the bootstrap distribution function in (3.16) can be strengthened to almost-sure convergence under additional smoothness conditions on the functional  $\mathbf{T}(\cdot)$ . However, the in-probability convergence of the bootstrap distribution is adequate for most statistical applications. Indeed, when  $p = 1$  (i.e.,  $\theta$  and  $\hat{\theta}_n$  are real-valued), (3.16) shows that a  $100(1 - \alpha)\%$  bootstrap percentile confidence interval for  $\theta$  of the form,

$$[\hat{\theta}_n - \lambda_n^{-d/2} \hat{q}_{n,1-\alpha/2}, \hat{\theta}_n - \lambda_n^{-d/2} \hat{q}_{n,\alpha/2}],$$

attains the nominal coverage probability  $(1 - \alpha)$  asymptotically, where  $\hat{q}_{n,\alpha}$  denotes the quantile of  $P_*(\lambda_n^{d/2}(\theta_n^* - \tilde{\theta}_n) \leq \cdot)$  and  $0 < \alpha < 1$ . In the multiparameter case, that is, for  $p \geq 2$ , one is typically interested in confidence regions of more general form than those defined through half-spaces. Hence, (3.15) is more useful than (3.16) to justify asymptotic validity of bootstrap confidence regions in the multiparameter setting.

## 4 Proofs

### 4.1 Proof of Theorem 3.1

Since the marginal CDF  $G(\cdot)$  is continuous by assumption (A.1), by standard arguments (cf. Pollard, 1984, pp.155), it is sufficient to show that the rescaled process  $\tilde{\xi}_n(u) \equiv \xi_n(G^{-1}(u))$  converges in distribution to  $\tilde{\mathcal{W}}(u) \equiv \mathcal{W}(G^{-1}(u))$ ;  $u \in [0, 1]$ , as a random element of  $D[0, 1]$ , where  $D[0, 1]$  is the space of all right-continuous functions on  $[0, 1]$  with left-hand limits equipped with the Skorohod  $J_1$ -topology. We use Theorem 15.1 in Billingsley (1968) to prove the functional CLT of  $\tilde{\xi}_n(\cdot)$

by first showing weak convergence of its finite-dimensional distributions, and then establishing the tightness of  $\tilde{\xi}_n(\cdot)$  and almost-sure (a.s.) continuity of the sample paths of  $\tilde{W}(\cdot)$ .

To show weak convergence of its finite-dimensional distributions, we rewrite  $\tilde{\xi}_n(\cdot)$  as,

$$\tilde{\xi}_n(u) = \lambda_n^{d/2} N_n^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} X(u; \mathbf{s}),$$

where  $X(u; \mathbf{s}) \equiv I(Z(\mathbf{s}) \leq G^{-1}(u)) - u$ ;  $\mathbf{s} \in \mathbb{R}^d$ ,  $u \in [0, 1]$ . Under the stationarity assumption (A.1) of the r.f.  $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ , the r.f.  $\{X(u; \mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$  is stationary with mean  $\mathbb{E}(X(u; \mathbf{0})) = 0$  and  $\mathbb{P}(|X(u; \mathbf{0})| \leq 1) = 1$ . Note also that  $h_n^{d/2} N_n^{-1/2} \sim |R_0|^{1/2} \lambda_n^{d/2} N_n^{-1}$ . Hence, by Proposition 4.1 and Theorem 4.1 of Lahiri (2000), for a given  $u \in [0, 1]$ ,  $\tilde{\xi}_n(u) \xrightarrow{\mathcal{D}} N(0, |R_0|^{-1} \int_{\mathbb{R}^d} \tilde{\sigma}(u, u; \mathbf{s}) d\mathbf{s})$ , where  $\xrightarrow{\mathcal{D}}$  denotes the weak convergence of random variables as  $n \rightarrow \infty$ , and

$$\tilde{\sigma}(u, u; \mathbf{s}) = \mathbb{E}(X(u; \mathbf{0})X(u; \mathbf{s})) = \tilde{H}(u, u; \mathbf{s}) - u^2; \quad u \in [0, 1], \quad \mathbf{s} \in \mathbb{R}^d,$$

where  $\tilde{H}(u_1, u_2; \mathbf{s}) \equiv H(G^{-1}(u_1), G^{-1}(u_2); \mathbf{s})$  denotes the rescaled version of  $H(u_1, u_2; \mathbf{s})$  defined in (2.2).

By similar arguments and the Cramér-Wold device (cf. Billingsley, 1968, pp.48–49), for  $i, j \in \{1, \dots, m\}$  and  $m \in \mathbb{Z}^+$ , the covariance function for  $(X(u_1; \mathbf{s}), \dots, X(u_m; \mathbf{s}))'$  is  $\tilde{\sigma}(u_i, u_j; \mathbf{s}) = \mathbb{E}(X(u_i; \mathbf{0})X(u_j; \mathbf{s})) = \tilde{H}(u_i, u_j; \mathbf{s}) - u_i u_j$ , and for given  $u_1, \dots, u_m \in [0, 1]$ ,

$$\begin{aligned} (\tilde{\xi}_n(u_1), \dots, \tilde{\xi}_n(u_m))' &= \lambda_n^{d/2} N_n^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} (X(u_1; \mathbf{s}), \dots, X(u_m; \mathbf{s}))' \\ &\xrightarrow{\mathcal{D}} N\left(\mathbf{0}, \left[|R_0|^{-1} \int_{\mathbb{R}^d} \tilde{\sigma}(u_i, u_j; \mathbf{s}) d\mathbf{s}\right]_{i,j=1}^m\right), \end{aligned}$$

where  $\xrightarrow{\mathcal{D}}$  denotes the weak convergence of random vectors as  $n \rightarrow \infty$ . Hence the convergence of finite-dimensional distributions is verified.

For the tightness of  $\tilde{\xi}_n(\cdot)$  and a.s. continuity of the sample paths of  $\tilde{W}(\cdot)$ , it is sufficient to show that, for all  $\epsilon > 0$  and all  $\eta > 0$ , there exist a  $\delta \in (0, 1)$  and an  $n_0 \in \mathbb{Z}^+$  such that,

$$\mathbb{P}\left(\sup_{u_0 \leq u \leq (u_0 + \delta) \wedge 1} |\tilde{\xi}_n(u) - \tilde{\xi}_n(u_0)| \geq \epsilon\right) \leq \eta \delta, \quad (4.1)$$

for all  $n \geq n_0$  and  $u_0 \in [0, 1]$ . The proof is similar to that of Theorem 22.1 in Billingsley (1968).

Let  $\mathcal{L}_n \equiv \{\mathbf{l} \in \mathbb{Z}^d : (\mathbf{l} + [0, 1]^d) \cap R_n \neq \emptyset\}$  denote the index set of unit squares in  $R_n$ . Rewrite  $\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1)$  in terms of sums over unit squares as,

$$\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1) = \lambda_n^{-d/2} \sum_{\mathbf{l} \in \mathcal{L}_n} Y(u_1, u_2; \mathbf{l}),$$

where  $Y(u_1, u_2; \mathbf{l}) \equiv \lambda_n^d N_n^{-1} \sum_{\mathbf{s} \in (\mathbf{l} + (0, 1)^d) \cap \mathcal{S}_n} X(u_2; \mathbf{s}) - X(u_1; \mathbf{s})$ ;  $\mathbf{l} \in \mathcal{L}_n$ ,  $u_1, u_2 \in [0, 1]$ . For  $0 \leq u_1 < u_2 \leq 1$ , it can be easily verified that,

$$\mathbb{E}(Y(u_1, u_2; \mathbf{l}))^2 = \lambda_n^{2d} N_n^{-2} \mathbb{E}\left[\sum_{\mathbf{s} \in (\mathbf{l} + (0, 1)^d) \cap \mathcal{S}_n} X(u_2; \mathbf{s}) - X(u_1; \mathbf{s})\right]^2 \leq \text{const.}(u_2 - u_1),$$

for all  $\mathbf{l} \in \mathcal{L}_n$ . Now, by similar arguments as in the proof of Lemma 6.1 (Lahiri, 2000),

$$\mathbb{E}\left[\sum_{\mathbf{l} \in \mathcal{L}_n} Y(u_1, u_2; \mathbf{l})\right]^4 \leq \text{const.}\left(\lambda_n^{2d}(u_2 - u_1)^2 + \lambda_n^d(u_2 - u_1)\right).$$

Fix  $u_1 \in [0, 1]$ ,  $\epsilon \in (0, 1)$  and  $\eta > 0$ . Choose  $n_0$  such that, for all  $n \geq n_0$ ,  $\epsilon \lambda_n^{-d} \leq u_2 - u_1$ . Then,

$$\begin{aligned} \mathbb{E}(\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1))^4 &\leq \lambda_n^{-2d} \text{const.} \left( \lambda_n^{2d} (u_2 - u_1)^2 + \lambda_n^d (u_2 - u_1) \right) \\ &\leq \text{const.} \left( (u_2 - u_1)^2 \epsilon^{-1} + (u_2 - u_1)^2 \epsilon^{-1} \right) \\ &= \text{const.} \cdot 2 (u_2 - u_1)^2 \epsilon^{-1}. \end{aligned}$$

Let  $p \geq \epsilon \lambda_n^{-d}$ . By Theorem 12.2 of Billingsley (1968),

$$\mathbb{P} \left( \max_{1 \leq i \leq m} |\tilde{\xi}_n(u_1 + ip) - \tilde{\xi}_n(u_1)| \geq \epsilon \right) \leq \text{const.} m^2 p^2 (2\epsilon^5)^{-1}. \quad (4.2)$$

Now, for  $u_1 \leq u_2 \leq u_1 + p$ , (as in (22.17) of Billingsley (1968)),

$$|\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1)| \leq |\tilde{\xi}_n(u_1 + p) - \tilde{\xi}_n(u_1)| + p \lambda_n^{d/2},$$

and hence,

$$\sup_{u_1 \leq u_2 \leq u_1 + mp} |\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1)| \leq 3 \max_{1 \leq i \leq m} |\tilde{\xi}_n(u_1 + ip) - \tilde{\xi}_n(u_1)| + p \lambda_n^{d/2}. \quad (4.3)$$

Finally, with the choice of  $\epsilon \lambda_n^{-d} \leq p < \epsilon \lambda_n^{-d/2}$ , and by (4.2) and (4.3), for every  $n \geq n_0$ ,

$$\begin{aligned} \mathbb{P}(\sup_{u_1 \leq u_2 \leq u_1 + mp} |\tilde{\xi}_n(u_2) - \tilde{\xi}_n(u_1)| \geq 4\epsilon) &\leq \mathbb{P}(\max_{1 \leq i \leq m} |\tilde{\xi}_n(u_1 + ip) - \tilde{\xi}_n(u_1)| \geq \epsilon) \\ &\leq \text{const.} m^2 p^2 \epsilon^{-5}. \end{aligned} \quad (4.4)$$

In (4.4), we choose  $\delta$  such that  $\text{const.} \epsilon^{-5} \delta \leq \eta$ , and  $m$  such that  $mp = \delta$ . We can find such  $m$  by choosing  $\delta \epsilon^{-1} \lambda_n^{d/2} < m \leq \delta \epsilon^{-1} \lambda_n^d$ . Hence  $\text{const.} m^2 p^2 \epsilon^{-5} \leq \eta \delta$  and (4.1) holds.  $\square$

## 4.2 Proof of Theorem 3.2

### Lemmas for Finite Dimensional Distributions

Let  $\tilde{\xi}_n^*(\cdot) \equiv \xi_n^*(G^{-1}(\cdot))$ , where  $\xi_n^*$  is defined in (2.7). For notational convenience, we abbreviate  $\mathcal{B}_{n,\mathbf{k}}$  to  $\mathcal{B}_{\mathbf{k}}$ ,  $\mathcal{B}_{n,\mathbf{k}}^0$  to  $\mathcal{B}_{\mathbf{k}}^0$ , and  $\mathcal{B}_{n,\mathbf{k}}^*$  to  $\mathcal{B}_{\mathbf{k}}^*$ , where  $\mathcal{B}_{n,\mathbf{k}}$ ,  $\mathcal{B}_{n,\mathbf{k}}^0$ , and  $\mathcal{B}_{n,\mathbf{k}}^*$  are defined in Section 2.3. Denote the empirical process at the block level by,

$$X_{\mathbf{k}}^*(u) \equiv |\mathcal{B}_{\mathbf{0}}|^{-1} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^*} |R_{\mathbf{0}}|^{-1/2} \lambda_l^{d/2} [I(Z(\mathbf{s}) \leq G^{-1}(u)) - u]; \quad u \in [0, 1], \mathbf{k} \in \mathcal{K}_n. \quad (4.5)$$

Then,  $\tilde{\xi}_n^*(u) = \lambda_n^{d/2} \lambda_l^{-d/2} |R_{\mathbf{0}}|^{1/2} |\mathcal{B}_{\mathbf{0}}| (\sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{\mathbf{k}}|)^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} [X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))]; \quad u \in [0, 1]$ .

Throughout this section, we shall use the fact that, given an  $a > 0$ ,

$$\mathbb{E}_*(X_{\mathbf{k}}^*(u)^a) = |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} X_{\mathbf{k}\mathbf{t}}(u)^a, \quad (4.6)$$

where  $X_{\mathbf{k}\mathbf{t}}(u) = |\mathcal{B}_{\mathbf{0}}|^{-1} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^0 + \mathbf{t}h_n} |R_{\mathbf{0}}|^{-1/2} \lambda_l^{d/2} [I(Z(\mathbf{s}) \leq G^{-1}(u)) - u]; \quad u \in [0, 1], \mathbf{k} \in \mathcal{K}_n, \mathbf{t} \in \mathcal{T}_n$ .

Moreover, we partition the index set  $\mathcal{K}_n$  into two subsets according to whether the block  $R_{n,\mathbf{k}}$ ;  $\mathbf{k} \in \mathcal{K}_n$  intersects the boundary or not:  $\mathcal{K}_{n,1} \equiv \{\mathbf{k} \in \mathbb{Z}^d : (\mathbf{k} + [0, 1]^d) \lambda_l \subset R_n\}$  and  $\mathcal{K}_{n,2} \equiv \{\mathbf{k} \in \mathbb{Z}^d : (\mathbf{k} + [0, 1]^d) \lambda_l \cap R_n \neq \emptyset, (\mathbf{k} + [0, 1]^d) \lambda_l \cap R_n^c \neq \emptyset\}$ , respectively.

**Lemma 4.1** Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1), (A.2), (A.4), and (A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,

$$\text{Var}_*(X_{\mathbf{0}}^*(u)) \rightarrow \tilde{\sigma}(u, u) \quad \text{a.s.},$$

where  $\tilde{\sigma}(u, u) \equiv |R_0|^{-1} \int_{\mathbb{R}^d} [\tilde{H}(u, u; \mathbf{s}) - u^2] d\mathbf{s}$ .

**Proof:** Since  $\text{Var}_*(X_{\mathbf{0}}^*(u)) = \text{E}_*(X_{\mathbf{0}}^*(u)^2) - [\text{E}_*(X_{\mathbf{0}}^*(u))]^2$ . We show that

$$\text{E}_*(X_{\mathbf{0}}^*(u)^2) \rightarrow \tilde{\sigma}(u, u) \quad \text{a.s.} \quad (4.7)$$

$$\text{E}_*(X_{\mathbf{0}}^*(u)) \rightarrow 0 \quad \text{a.s.} \quad (4.8)$$

To show (4.7), note that by the stationarity of  $Z(\cdot)$  and the same arguments as in the proof of Theorem 3.1, we obtain,

$$\text{E}(\text{E}_*(X_{\mathbf{0}}^*(u)^2)) = |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} \text{E}(X_{\mathbf{0}\mathbf{t}}(u)^2) = \text{E}(X_{\mathbf{0}\mathbf{0}}(u)^2) \rightarrow \tilde{\sigma}(u, u).$$

Further, we show that  $\text{E}[\text{E}_*(X_{\mathbf{0}}^*(u)^2) - \text{E}(X_{\mathbf{0}\mathbf{0}}(u)^2)]^{2r}$  is summable over  $n$  with  $r = r_0$ . To do so, we partition  $R_n$  into blocks of side length  $2\lambda_l$  and denote the corresponding index set by  $\mathcal{J}_n \equiv \{\mathbf{j} \in \mathbb{Z}^d : (\mathbf{j} + [0, 1)^d)2\lambda_l \cap R_n \neq \emptyset\}$ . Next, we group these blocks into  $2^d$  classes, each with a corresponding index subset defined as,  $\mathcal{J}_{n,\mathbf{p}} \equiv \{\mathbf{j} \in \mathcal{J}_n : (\mathbf{j} - \mathbf{p})/2 \in \mathbb{Z}^d\}$ ;  $\mathbf{p} \in \{0, 1\}^d$ , so that the distance between two blocks within each class is at least  $\lambda_l$ -distance apart. Hence, we obtain,

$$\text{E}_*[X_{\mathbf{0}}^*(u)^2 - \text{E}(X_{\mathbf{0}\mathbf{0}}(u)^2)] = |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} [X_{\mathbf{0}\mathbf{t}}(u)^2 - \text{E}(X_{\mathbf{0}\mathbf{0}}(u)^2)] = |\mathcal{T}_n|^{-1} \sum_{\mathbf{p}} \sum_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} V_{\mathbf{0}}(\mathbf{j}),$$

where  $V_{\mathbf{0}}(\mathbf{j}) \equiv \sum_{\mathbf{t} \in \mathcal{T}_n, h_n \mathbf{t} \in (\mathbf{j} + [0, 1)^d)2\lambda_l \cap R_n} [X_{\mathbf{0}\mathbf{t}}(u)^2 - \text{E}(X_{\mathbf{0}\mathbf{0}}(u)^2)]$  denotes the sum over the block  $(\mathbf{j} + [0, 1)^d)2\lambda_l \cap R_n$ ;  $\mathbf{j} \in \mathcal{J}_n$ . Now, it suffices to show that  $\text{E}[|\mathcal{T}_n|^{-1} \sum_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} V_{\mathbf{0}}(\mathbf{j})]^{2r}$  is summable for every  $\mathbf{p} \in \{0, 1\}^d$ . Note that  $V_{\mathbf{0}}(\mathbf{j}_1)$  and  $V_{\mathbf{0}}(\mathbf{j}_2)$  are separated by a distance no less than  $(\text{const.} \|\mathbf{j}_1 - \mathbf{j}_2\|_{\infty} \lambda_l)$  for all  $\mathbf{j}_1 \neq \mathbf{j}_2$ , where  $\|\cdot\|_{\infty}$  is the  $\ell^{\infty}$ -norm on  $\mathbb{R}^d$  defined by  $\|\mathbf{x}\|_{\infty} = \max\{|x_i| : 1 \leq i \leq d\}$ ;  $\mathbf{x} = (x_1, \dots, x_d)' \in \mathbb{R}^d$ . Further, with  $\epsilon = r$ , since  $\tau > 3d(2r - 1)$ ,

$$\sum_{i=1}^{\infty} (i+1)^{d(2r-1)-1} \alpha(i\lambda_l; \lambda_l^d)^{\epsilon/(2r+\epsilon)} \leq \text{const.} \sum_{i=1}^{\infty} i^{d(2r-1)-1} i^{-\tau/3} [\lambda_l^{-\tau+\theta d}]^{1/3} < \infty. \quad (4.9)$$

Hence, by Theorem 1.4.1.1 in Doukhan (1994), the Minkowski's Inequality, and the stationarity of  $Z(\cdot)$ , we obtain,

$$\begin{aligned} \text{E} \left[ \sum_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} V_{\mathbf{0}}(\mathbf{j}) \right]^{2r} &\leq \text{const.} |\mathcal{J}_{n,\mathbf{p}}|^r \max_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} \|V_{\mathbf{0}}(\mathbf{j})\|_{3r}^{2r} \\ &\leq \text{const.} |\mathcal{J}_{n,\mathbf{p}}|^r (\lambda_l^d h_n^{-d})^{2r} \|X_{\mathbf{0}\mathbf{0}}(u)^2\|_{3r}^{2r} \\ &\leq \text{const.} |\mathcal{J}_{n,\mathbf{p}}|^r (\lambda_l^d h_n^{-d})^{2r} |\mathcal{B}_{\mathbf{0}}|^{-4r} \lambda_l^{2dr} \left\| \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}}} I(Z(\mathbf{s}) \leq G^{-1}(u)) - u \right\|_{6r}^{4r}. \end{aligned} \quad (4.10)$$

To find the order of  $\left\| \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}}} I(Z(\mathbf{s}) \leq G^{-1}(u)) - u \right\|_{6r}$ , we group the sampling sites within unit blocks and denote the corresponding index set by  $\mathcal{U}_{\mathbf{0}} \equiv \{\mathbf{i} \in \mathbb{Z}^d : \mathcal{B}_{\mathbf{0}} \cap (\mathbf{i} + [0, 1)^d) \neq \emptyset\}$ . Since

$\tau > d(6r - 1)$  (for  $r = r_0$ ), there exists a large positive constant  $\epsilon_0$  such that  $\tau\epsilon_0(6r + \epsilon_0)^{-1} > d(6r - 1)$ . Hence,

$$\sum_{i=1}^{\infty} (i+1)^{d(6r-1)-1} \alpha(i; 1)^{\epsilon_0/(6r+\epsilon_0)} < \infty. \quad (4.11)$$

Now applying Theorem 1.4.1.1 in Doukhan (1994) again, we have,

$$\begin{aligned} \mathbb{E} \left[ \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}}} I(Z(\mathbf{s}) \leq G^{-1}(u)) - u \right]^{6r} &= \mathbb{E} \left[ \sum_{\mathbf{i} \in \mathcal{U}_{\mathbf{0}}} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}} \cap (\mathbf{i} + [0,1]^d)} I(Z(\mathbf{s}) \leq G^{-1}(u)) - u \right]^{6r} \\ &\leq \text{const.} |\mathcal{U}_{\mathbf{0}}|^{3r} \max_{\mathbf{i} \in \mathcal{U}_{\mathbf{0}}} \left\| \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}} \cap (\mathbf{i} + [0,1]^d)} I(Z(\mathbf{s}) \leq G^{-1}(u)) - u \right\|_{6r+\epsilon_0}^{6r} \\ &\leq \text{const.} |\mathcal{U}_{\mathbf{0}}|^{3r} (h_n^{-d})^{6r}. \end{aligned} \quad (4.12)$$

Hence, by (4.10) and (4.12),

$$\begin{aligned} \mathbb{E}[|\mathcal{T}_n|^{-1} \sum_{\mathbf{j} \in \mathcal{J}_n, \mathbf{p}} V_{\mathbf{0}}(\mathbf{j})]^{2r} &\leq \text{const.} |\mathcal{T}_n|^{-2r} |\mathcal{J}_n, \mathbf{p}|^r (\lambda_l^d h_n^{-d})^{2r} |\mathcal{B}_{\mathbf{0}}|^{-4r} \lambda_l^{2dr} |\mathcal{U}_{\mathbf{0}}|^{2r} (h_n^{-d})^{4r} \\ &\leq \text{const.} \lambda_l^{dr} \lambda_n^{-dr}. \end{aligned} \quad (4.13)$$

By the assumption (A.5),  $\mathbb{E}[|\mathcal{T}_n|^{-1} \sum_{\mathbf{j} \in \mathcal{J}_n, \mathbf{p}} V_{\mathbf{0}}(\mathbf{j})]^{2r}$  is summable over  $n$  with  $r = r_0$ . Hence (4.7) holds.

For proving (4.8), we group the sampling sites within unit cubes. Recall that the corresponding index set is denoted by  $\mathcal{L}_n = \{\mathbf{l} \in \mathbb{Z}^d : (\mathbf{l} + [0,1]^d) \cap R_n \neq \emptyset\}$ . Note that,

$$\begin{aligned} \mathbb{E}_*(X_{\mathbf{0}}^*(u)) &= |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} X_{\mathbf{0}\mathbf{t}}(u) \\ &= |\mathcal{T}_n|^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} \omega_n(\mathbf{s}) |R_{\mathbf{0}}|^{-1/2} \lambda_l^{d/2} [I(Z(\mathbf{s}) \leq G^{-1}(u)) - u] \\ &= |\mathcal{T}_n|^{-1} \sum_{\mathbf{l} \in \mathcal{L}_n} Y(\mathbf{l}), \end{aligned}$$

where the constants  $\omega_n(\mathbf{s}) \in [0, 1]$ , for all  $\mathbf{s} \in \mathcal{S}_n$ , and  $Y(\mathbf{l}) \equiv \sum_{\mathbf{s} \in (\mathbf{l} + (0,1)^d) \cap \mathcal{S}_n} \omega_n(\mathbf{s}) |R_{\mathbf{0}}|^{-1/2} \lambda_l^{d/2} [I(Z(\mathbf{s}) \leq G^{-1}(u)) - u]$  denotes the sum over the unit cube  $(\mathbf{l} + (0,1)^d)$ ;  $\mathbf{l} \in \mathcal{L}_n$ . By the stationarity of  $Z(\cdot)$ ,  $\mathbb{E}(\mathbb{E}_*(X_{\mathbf{0}}^*(u))) = 0$ . Also by (A.2),  $\tau > 3d(2r - 1)$ , and Theorem 1.4.1.1 in Doukhan (1994),

$$\begin{aligned} \mathbb{E}[\mathbb{E}_*(X_{\mathbf{0}}^*(u))]^{2r} &= \mathbb{E} \left[ |\mathcal{T}_n|^{-1} \sum_{\mathbf{l} \in \mathcal{L}_n} Y(\mathbf{l}) \right]^{2r} \\ &\leq \text{const.} |\mathcal{T}_n|^{-2r} |\mathcal{L}_n|^r \max_{\mathbf{l} \in \mathcal{L}_n} \|Y(\mathbf{l})\|_{3r}^{2r} \\ &\leq \text{const.} |\mathcal{T}_n|^{-2r} |\mathcal{L}_n|^r \max_{\mathbf{l} \in \mathcal{L}_n} \left\| \sum_{\mathbf{s} \in (\mathbf{l} + [0,1]^d) \cap \mathcal{S}_n} \lambda_l^{d/2} (I(Z(\mathbf{s}) \leq G^{-1}(u)) - u) \right\|_{3r}^{2r} \\ &\leq \text{const.} |\mathcal{T}_n|^{-2r} |\mathcal{L}_n|^r \lambda_l^{dr} (h_n^{-d})^{2r} \\ &\leq \text{const.} \lambda_l^{dr} \lambda_n^{-dr}, \end{aligned}$$

which is summable over  $n$  by assumption (A.5). Hence (4.8) holds.  $\square$

**Lemma 4.2** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \text{Var}_*(X_{\mathbf{k}}^*(u)) \rightarrow 0 \quad \text{a.s.}$$

**Proof:** Since  $\text{Var}_*(X_{\mathbf{k}}^*(u)) \leq \text{E}_*(X_{\mathbf{k}}^*(u))^2$ , it suffices to show that,

$$A_{n1} \equiv |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \text{E}_*(X_{\mathbf{k}}^*(u))^2 \rightarrow 0 \quad \text{a.s.}$$

Note that  $A_{n1} = |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} X_{\mathbf{k}\mathbf{t}}(u)^2$ . Hence, by the stationarity of  $Z(\cdot)$ , (A.3), and (A.5),

$$\text{E}(A_{n1}) = |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \text{E}(X_{\mathbf{k}\mathbf{0}}(u)^2) \leq \text{const.} |\mathcal{K}_n|^{-1} |\mathcal{K}_{n,2}| = \mathcal{O}(\lambda_l/\lambda_n) \rightarrow 0.$$

Next, using the arguments leading to (4.13) in the proof of Lemma 4.1 and the fact that  $|\mathcal{B}_{\mathbf{k}}| \leq |\mathcal{B}_{\mathbf{0}}|$  for all  $\mathbf{k} \in \mathcal{K}_{n,2}$ , one can show that, with  $r = r_0$ ,

$$\begin{aligned} \text{E}[A_{n1} - \text{E}(A_{n1})]^{2r} &\leq \text{const.} (|\mathcal{K}_n|^{-1} |\mathcal{K}_{n,2}|)^{2r} \max_{\mathbf{k} \in \mathcal{K}_{n,2}} |\mathcal{T}_n|^{-2r} \text{E} \left[ \sum_{\mathbf{t} \in \mathcal{T}_n} (X_{\mathbf{k}\mathbf{t}}(u)^2 - \text{E}(X_{\mathbf{k}\mathbf{t}}(u)^2))^{2r} \right] \\ &\leq \text{const.} (\lambda_l/\lambda_n)^{2r} \lambda_l^{dr} \lambda_n^{-dr}. \end{aligned}$$

Hence, Lemma 4.2 follows.  $\square$

**Lemma 4.3** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \text{Var}_*(X_{\mathbf{k}}^*(u)) \rightarrow \tilde{\sigma}(u, u) \quad \text{a.s.}$$

**Proof:** Rewrite,

$$\begin{aligned} |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \text{Var}_*(X_{\mathbf{k}}^*(u)) &= |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,1}} \text{Var}_*(X_{\mathbf{k}}^*(u)) + |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \text{Var}_*(X_{\mathbf{k}}^*(u)) \\ &\equiv A_{n2} + A_{n3}. \end{aligned}$$

Note that for  $\mathbf{k} \in \mathcal{K}_{n,1}$ ,  $X_{\mathbf{k}}^*(u)$  are i.i.d, and  $|\mathcal{K}_{n,1}| \sim |\mathcal{K}_n|$ . By Lemma 4.1,  $\text{Var}_*(X_{\mathbf{0}}^*(u)) \rightarrow \tilde{\sigma}(u, u)$  a.s. Hence,  $A_{n2} \rightarrow \tilde{\sigma}(u, u)$  a.s. By Lemma 4.2,  $A_{n3} \rightarrow 0$  a.s. Hence, the result of the lemma holds.  $\square$

**Lemma 4.4** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1), (A.2), (A.4), and (A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$\text{E}_*(X_{\mathbf{0}}^*(u)^4) \rightarrow 3\tilde{\sigma}(u, u)^2 \quad \text{a.s.}$$

**Proof:** By the stationarity of  $Z(\cdot)$  and same arguments as in the proof of Theorem 3.1, we obtain,

$$\mathbb{E}(\mathbb{E}_*(X_{\mathbf{0}}^*(u)^4)) = |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} \mathbb{E}(X_{\mathbf{0}\mathbf{t}}(u)^4) = \mathbb{E}(X_{\mathbf{0}\mathbf{0}}(u)^4) \rightarrow 3\tilde{\sigma}(u, u)^2.$$

Furthermore,

$$\mathbb{E}_*[X_{\mathbf{0}}^*(u)^4 - \mathbb{E}(X_{\mathbf{0}\mathbf{0}}(u)^4)] = |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} [X_{\mathbf{0}\mathbf{t}}(u)^4 - \mathbb{E}(X_{\mathbf{0}\mathbf{0}}(u)^4)] = |\mathcal{T}_n|^{-1} \sum_{\mathbf{p}} \sum_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} W_{\mathbf{0}}(\mathbf{j}),$$

where  $W_{\mathbf{0}}(\mathbf{j}) \equiv \sum_{\mathbf{t} \in \mathcal{T}_n, h_n \mathbf{t} \in (\mathbf{j} + [0,1]^d)_{2\lambda_l} \cap R_n} [X_{\mathbf{0}\mathbf{t}}(u)^4 - \mathbb{E}(X_{\mathbf{0}\mathbf{0}}(u)^4)]$ ;  $\mathbf{j} \in \mathcal{J}_n$ . As in the derivation of (4.13), it can be shown that, with  $r = r_0$ ,

$$\mathbb{E} \left[ \left| |\mathcal{T}_n|^{-1} \sum_{\mathbf{j} \in \mathcal{J}_{n,\mathbf{p}}} W_{\mathbf{0}}(\mathbf{j}) \right| \right]^{2r} \leq \text{const.} \lambda_l^{dr} \lambda_n^{-dr},$$

for every  $\mathbf{p} \in \{0, 1\}^d$ . Hence, the lemma follows.  $\square$

**Lemma 4.5** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \mathbb{E}_* |X_{\mathbf{k}}^*(u)|^3 \rightarrow 0 \quad \text{a.s.}$$

**Proof:** Similar to the proof of Lemma 4.2. We omit the details.  $\square$

**Lemma 4.6** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ ,*

$$|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3 \leq \text{const.} \quad \text{a.s.}$$

**Proof:** Rewrite,

$$\begin{aligned} & |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3 \\ &= |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,1}} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3 + |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3 \\ &\leq \text{const.} (|\mathcal{K}_{n,1}| |\mathcal{K}_n|^{-1} [\mathbb{E}_*(X_{\mathbf{0}}^*(u)^4)]^{3/4} + |\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_{n,2}} \mathbb{E}_* |X_{\mathbf{k}}^*(u)|^3) \\ &\equiv \text{const.} (|\mathcal{K}_{n,1}| |\mathcal{K}_n|^{-1} A_{n4}^{3/4} + A_{n5}). \end{aligned}$$

By Lemma 4.4,  $A_{n4} \rightarrow 3\tilde{\sigma}(u, u)^2$  a.s. and by Lemma 4.5,  $A_{n5} \rightarrow 0$  a.s. Since  $|\mathcal{K}_{n,1}|/|\mathcal{K}_n| \sim 1$ , the result of the lemma holds.  $\square$

**Lemma 4.7** Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \tau_1^0$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ , for any given  $m \in \mathbf{Z}^+$  as  $n \rightarrow \infty$ ,

$$(\tilde{\xi}_n^*(u_1), \dots, \tilde{\xi}_n^*(u_m))' \xrightarrow{\mathcal{D}} (\tilde{\mathcal{W}}(u_1), \dots, \tilde{\mathcal{W}}(u_m))' \quad \text{a.s.},$$

for all  $u_1, \dots, u_m \in [0, 1]$ , where  $\xrightarrow{\mathcal{D}}$  denotes weak convergence of random vectors in  $\mathbb{R}^m$ . That is, there exists a set  $A \in \mathcal{F}$  with  $\mathbb{P}(A) = 1$  such that for all  $\omega \in A$ , the finite dimensional distributions of  $\tilde{\xi}_n^*(\cdot; \omega)$  converges weakly to those of  $\tilde{\mathcal{W}}(\cdot)$ , as  $n \rightarrow \infty$ .

**Proof:** It suffices to show that, for any real numbers  $a_1, \dots, a_m \in \mathbb{R}$ , and  $u_1, \dots, u_m \in [0, 1]$ ,

$$\sum_{i=1}^m a_i \tilde{\xi}_n^*(u_i) \xrightarrow{\mathcal{D}} \sum_{i=1}^m a_i \tilde{\mathcal{W}}(u_i) \quad \text{a.s.},$$

where  $\xrightarrow{\mathcal{D}}$  denotes weak convergence of random variables. Without loss of generality, we show that for  $m = 1$  and a given  $u \in [0, 1]$ ,  $\tilde{\xi}_n^*(u) \xrightarrow{\mathcal{D}} \tilde{\mathcal{W}}(u)$ , a.s.

By Berry-Essen Lemma (cf. Shao and Tu, 1995, pp.451), Lemma 4.3, Lemma 4.6, and assumption (A.5), we obtain,

$$\begin{aligned} & \sup_x |\mathbb{P}_*(\text{Var}_*(\tilde{\xi}_n^*(u))^{-1/2} \tilde{\xi}_n^*(u) \leq x) - \Phi(x)| \\ & \leq \text{const.} \left[ \sum_{\mathbf{k} \in \mathcal{K}_n} \text{Var}_*(X_{\mathbf{k}}^*(u)) \right]^{-3/2} \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3 \\ & = \text{const.} |\mathcal{K}_n|^{-1/2} \frac{|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_* |X_{\mathbf{k}}^*(u) - \mathbb{E}_*(X_{\mathbf{k}}^*(u))|^3}{[|\mathcal{K}_n|^{-1} \sum_{\mathbf{k} \in \mathcal{K}_n} \text{Var}_*(X_{\mathbf{k}}^*(u))]^{3/2}} \\ & \leq \text{const.} \lambda_i^{d/2} / \lambda_n^{d/2} \rightarrow 0 \quad \text{a.s.} \quad \square \end{aligned}$$

### Lemmas for Tightness

Now, fix  $\epsilon > 0$ ,  $\eta > 0$ , and  $u_0 \in (0, 1)$ . Let  $\delta \in (0, 1)$  denote a constant to be specified later. We divide the interval  $[u_0, u_0 + \delta]$  into subintervals  $(u_i, u_{i+1}]$  such that  $u_i = u_0 + i\epsilon/(6\lambda_n^{d/2})$ ;  $i = 0, 1, \dots, m_n$ , where  $m_n \equiv \lceil \delta(6\lambda_n^{d/2})/\epsilon \rceil$  denotes the largest integer not exceeding  $\delta(6\lambda_n^{d/2})/\epsilon$ .

Further, with  $\Delta u_i \equiv u_{i+1} - u_i$ , we let,

$$\tilde{X}_{\mathbf{k}}^*(u_i, u_{i+1}) \equiv |\mathcal{B}_{\mathbf{0}}|^{-1} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^*} [I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i]. \quad (4.14)$$

Then,

$$\tilde{\xi}_n^*(u_{i+1}) - \tilde{\xi}_n^*(u_i) = \lambda_n^{d/2} \left( \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{\mathbf{k}}| \right)^{-1} |\mathcal{B}_{\mathbf{0}}| \sum_{\mathbf{k} \in \mathcal{K}_n} [\tilde{X}_{\mathbf{k}}^*(u_i, u_{i+1}) - \mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(u_i, u_{i+1}))]. \quad (4.15)$$

Also, let  $\tilde{F}_n^*(\cdot) \equiv F_n^*(G^{-1}(\cdot))$ . In the rest of the section, we shall use the fact that, given an  $a > 0$ ,

$$\mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(u_i, u_{i+1})^a) = |\mathcal{I}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{I}_n} \tilde{X}_{\mathbf{k}\mathbf{t}}(u_i, u_{i+1})^a, \quad (4.16)$$

where  $\tilde{X}_{\mathbf{k}\mathbf{t}}(u_i, u_{i+1}) \equiv |\mathcal{B}_{\mathbf{0}}|^{-1} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^{\mathbf{0}_+ \mathbf{t}_{h_n}}} [I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i]$ . Henceforth, we abbreviate  $f(u_{i+1}) - f(u_i)$  to  $f(\psi_i)$ , for any real-valued function  $f$ .

**Lemma 4.8** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \max\{35d, \tau_1^0, \tau_2^0, \tau_3^0\}$ ,  $\theta < \tau/d$ . Then, as  $n \rightarrow \infty$ , for any given  $u_0 \in [0, 1]$ , for every  $\epsilon > 0$ ,  $\eta > 0$ , there exist a  $\delta \in (0, 1)$  and an  $n_0 \in \mathbf{Z}^+$ , such that for all  $n \geq n_0$ ,*

$$P_*\left(\max_{0 \leq i \leq m_n - 1} |\tilde{\xi}_n^*(\psi_i)| \geq \epsilon/6\right) \leq \eta\delta/2 \quad \text{a.s.}$$

**Proof:** We apply similar arguments as in the proof of Lemma 7 and 8 of Bühlmann (1994). The main task here is to derive the following inequality: There exist a set  $A \in \mathcal{F}$  with  $P(A) = 1$ , an integer  $r \geq 1$ , and a real number  $\zeta > 1$  such that for any  $\omega \in A$ ,

$$E_*[\tilde{\xi}_n^*(\psi_i)]^{2r}(\omega) \leq \text{const.}(\Delta u_i)^\zeta, \quad (4.17)$$

uniformly in  $i \in \{0, 1, \dots, m_n - 1\}$ , whenever  $n \geq N(\omega)$  for some  $N(\omega) \geq 1$ .

Set  $r = 2$ . Then by (4.15) and Rosenthal's Inequality (cf. Theorem 2.12, Hall and Heyde (1981)),

$$\begin{aligned} E_*|\tilde{\xi}_n^*(\psi_i)|^4 &\leq \text{const.}\lambda_n^{2d}|\mathcal{K}_n|^{-4} \left[ \sum_{\mathbf{k} \in \mathcal{K}_n} E_*|\tilde{X}_{\mathbf{k}}^*(\psi_i)|^4 + \left( \sum_{\mathbf{k} \in \mathcal{K}_n} E_*|\tilde{X}_{\mathbf{k}}^*(\psi_i)|^2 \right)^2 \right] \\ &\equiv \text{const.}\lambda_n^{2d}|\mathcal{K}_n|^{-4} \left[ \sum_{\mathbf{k} \in \mathcal{K}_n} A_i(\mathbf{k}) + \left( \sum_{\mathbf{k} \in \mathcal{K}_n} Q_i(\mathbf{k}) \right)^2 \right], \quad \text{say.} \end{aligned} \quad (4.18)$$

We shall show that there exist a  $\zeta > 1$  and integers  $r_1 \geq 1$ ,  $r_2 \geq 1$  such that,

$$I_0 \equiv \max_{0 \leq i \leq m_n - 1} \lambda_n^d |\mathcal{K}_n|^{-2} \sum_{\mathbf{k} \in \mathcal{K}_n} E(Q_i(\mathbf{k})) \leq \text{const.}\lambda_n^{-\zeta d/4}, \quad (4.19)$$

$$I_1 \equiv \max_{0 \leq i \leq m_n - 1} \lambda_n^{2d} |\mathcal{K}_n|^{-4} \sum_{\mathbf{k} \in \mathcal{K}_n} E(A_i(\mathbf{k})) \leq \text{const.}\lambda_n^{-\zeta d/2}, \quad (4.20)$$

$$I_2 \equiv \max_{0 \leq i \leq m_n - 1} E \left[ \lambda_n^{\zeta d/4} \lambda_n^d |\mathcal{K}_n|^{-2} \sum_{\mathbf{k} \in \mathcal{K}_n} (Q_i(\mathbf{k}) - E(Q_i(\mathbf{k}))) \right]^{2r_1} \leq \text{const.}\lambda_n^{-d/2} \lambda_n^{-\zeta d}, \quad (4.21)$$

$$I_3 \equiv \max_{0 \leq i \leq m_n - 1} E \left[ \lambda_n^{\zeta d/2} \lambda_n^{2d} |\mathcal{K}_n|^{-4} \sum_{\mathbf{k} \in \mathcal{K}_n} (A_i(\mathbf{k}) - E(A_i(\mathbf{k}))) \right]^{2r_2} \leq \text{const.}\lambda_n^{-d/2} \lambda_n^{-\zeta d}. \quad (4.22)$$

Then, assuming that (4.19)–(4.22) hold for now, from (4.18), we have,

$$\begin{aligned} &\sum_{n=1}^{\infty} P(E_*|\tilde{\xi}_n^*(\psi_i)|^4 > \text{const.}(\Delta u_i)^\zeta : \text{ for some } i \in \{0, 1, \dots, m_n - 1\}) \\ &\leq \sum_{n=1}^{\infty} P(\lambda_n^{2d} |\mathcal{K}_n|^{-4} [ |\sum_{\mathbf{k} \in \mathcal{K}_n} A_i(\mathbf{k}) - E(A_i(\mathbf{k}))| + |\sum_{\mathbf{k} \in \mathcal{K}_n} Q_i(\mathbf{k}) - E(Q_i(\mathbf{k}))|^2 ] \\ &\quad > \text{const.}\lambda_n^{-\zeta d/2} \text{ for some } i \in \{0, 1, \dots, m_n - 1\}) \\ &\leq \text{const.} \sum_{n=1}^{\infty} \sum_{i=1}^{m_n - 1} E \left[ \lambda_n^{\zeta d/2} \lambda_n^{2d} |\mathcal{K}_n|^{-4} \sum_{\mathbf{k} \in \mathcal{K}_n} (A_i(\mathbf{k}) - E(A_i(\mathbf{k}))) \right]^{2r_1} \end{aligned}$$

$$\begin{aligned}
& + \text{const.} \sum_{n=1}^{\infty} \sum_{i=1}^{m_n-1} \mathbb{E} \left[ \lambda_n^{\zeta d/4} \lambda_n^d |\mathcal{K}_n|^{-2} \sum_{\mathbf{k} \in \mathcal{K}_n} (Q_i(\mathbf{k}) - \mathbb{E}(Q_i(\mathbf{k}))) \right]^{2r_2} \\
\leq & \text{const.} \sum_{n=1}^{\infty} \lambda_n^{-\zeta d} < \infty.
\end{aligned} \tag{4.23}$$

Hence, by the Borel-Cantelli Lemma, Lemma 4.8 follows from (4.23).

Now, we show (4.19)–(4.22) under the assumed mixing condition. By stationarity and the  $\alpha$ -mixing condition (4.9) (with  $r = 1$ ), for any  $\mathbf{k} \in \mathcal{K}_{n,1}$ ,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(\psi_i)^2)] &= \mathbb{E}[\mathbb{E}_*(\tilde{X}_{\mathbf{0}}^*(\psi_i)^2)] = \mathbb{E}(\tilde{X}_{\mathbf{0}\mathbf{0}}(\psi_i)^2) \\
&= |\mathcal{B}_{\mathbf{0}}|^{-2} \mathbb{E} \left( \sum_{\mathbf{i} \in \mathcal{U}_{\mathbf{0}}} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}} \cap (\mathbf{i} + [0,1]^d)} I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i \right)^2 \\
&\leq \text{const.} |\mathcal{B}_{\mathbf{0}}|^{-2} |\mathcal{U}_{\mathbf{0}}| \max_{\mathbf{i} \in \mathcal{U}_{\mathbf{0}}} \left\| \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{0}} \cap (\mathbf{i} + [0,1]^d)} I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i \right\|_3^2 \\
&\leq \text{const.} \lambda_l^{-d} \lambda_n^{-d/3},
\end{aligned} \tag{4.24}$$

uniformly in  $i \in \{0, 1, \dots, m_n - 1\}$ . By similar arguments, we obtain,

$$\max_{\mathbf{k} \in \mathcal{K}_{n,2}} \max_{0 \leq i \leq m_n - 1} \mathbb{E}[\mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(\psi_i)^2)] \leq \text{const.} \lambda_l^{-d} \lambda_n^{-d/3}. \tag{4.25}$$

This proves (4.19). Next, consider (4.20). Let  $\mathcal{U}_{\mathbf{k}} \equiv \{\mathbf{j} \in \mathbb{Z}^d : \mathcal{B}_{\mathbf{k}}^{\mathbf{0}} \cap (\mathbf{j} + [0,1]^d) \neq \emptyset\}$  and  $\tilde{Y}_i(\mathbf{j}) \equiv \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^{\mathbf{0}} \cap (\mathbf{j} + [0,1]^d)} I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i$ . Since  $\tau > 15d$ , by Theorem 1.4.1.1 of Doukhan (1994),

$$\begin{aligned}
\mathbb{E}(\mathbb{E}_*|\tilde{X}_{\mathbf{k}}^*(\psi_i)|^4) &= \mathbb{E}(|\tilde{X}_{\mathbf{k}\mathbf{0}}(\psi_i)|^4) = |\mathcal{B}_{\mathbf{0}}|^{-4} \mathbb{E} \left[ \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \tilde{Y}_i(\mathbf{j}) \right]^4 \\
&\leq \text{const.} |\mathcal{B}_{\mathbf{0}}|^{-4} \left[ \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_5^4 + \left( \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_3^2 \right)^2 \right] \\
&\leq \text{const.} \lambda_l^{-4d} h_n^{4d} [\lambda_l^d h_n^{-4d} |\Delta u_i|^{4/5} + (\lambda_l^d h_n^{-2d} |\Delta u_i|^{2/3})^2]
\end{aligned}$$

uniformly in  $i \in \{0, 1, \dots, m_n - 1\}$  and  $\mathbf{k} \in \mathcal{K}_n$ . Thus,

$$I_1 \leq \text{const.} \lambda_n^{2d} |\mathcal{K}_n|^{-3} [\lambda_l^{-3d} \lambda_n^{-2d/5} + \lambda_l^{-2d} \lambda_n^{-2d/3}] \leq \text{const.} [\lambda_n^{-d} \lambda_n^{-2d/5} + \lambda_l^d \lambda_n^{-d} \lambda_n^{-2d/3}],$$

and hence (4.20). To prove (4.21) and (4.22), define the variables,

$$\tilde{V}_{\mathbf{k}}^s(\mathbf{j}) \equiv \sum_{\mathbf{t} \in \mathcal{T}_n, h_n \mathbf{t} \in (\mathbf{j} + [0,1]^d) \cap 2\lambda_l \cap R_n} [\tilde{X}_{\mathbf{k}\mathbf{t}}(\psi_i)^{2s} - \mathbb{E}(\tilde{X}_{\mathbf{k}\mathbf{t}}(\psi_i)^{2s})],$$

where  $\mathbf{k} \in \mathcal{K}_n$ ,  $\mathbf{j} \in \mathcal{J}_n$ ,  $s = 1, 2$ . Then, using the partitioning by the sets  $\mathcal{J}_n, \mathbf{p}$ ,  $\mathbf{p} \in \{0, 1\}^d$ , as in the proof of Lemma 4.1, for any  $r \in \mathbb{Z}^+$ , and  $\epsilon > 0$ ,

$$\mathbb{E} \left[ \mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(\psi_i)^{2s}) - \mathbb{E}(\mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(\psi_i)^{2s})) \right]^{2r}$$

$$\begin{aligned}
&\leq \text{const.} |\mathcal{T}_n|^{-2r} \sum_{\mathbf{p} \in \{0,1\}^d} \mathbb{E} \left[ \sum_{\mathbf{j} \in \mathcal{J}_n, \mathbf{p}} \tilde{V}_{\mathbf{k}}^s(\mathbf{j}) \right]^{2r} \\
&\leq \text{const.} |\mathcal{T}_n|^{-2r} \sum_{\mathbf{p} \in \{0,1\}^d} \left[ \sum_{\mathbf{j} \in \mathcal{J}_n, \mathbf{p}} \|\tilde{V}_{\mathbf{k}}^s(\mathbf{j})\|_{2r+\epsilon}^{2r} + \left( \sum_{\mathbf{j} \in \mathcal{J}_n, \mathbf{p}} \|\tilde{V}_{\mathbf{k}}^s(\mathbf{j})\|_{2+\epsilon}^2 \right)^r \right] \\
&\leq \text{const.} |\mathcal{T}_n|^{-2r} \left[ (\lambda_n^d / \lambda_l^d) (\lambda_l^d h_n^{-d})^{2r} \|\tilde{X}_{\mathbf{k}\mathbf{0}}(\psi_i)^{2s}\|_{2r+\epsilon}^{2r} + \left( (\lambda_n^d / \lambda_l^d) (\lambda_l^d h_n^{-d})^2 \|\tilde{X}_{\mathbf{k}\mathbf{0}}(\psi_i)^{2s}\|_{2+\epsilon}^2 \right)^r \right],
\end{aligned} \tag{4.26}$$

uniformly in  $i \in \{0, 1, \dots, m_n - 1\}$ , provided  $\tau > d(2r - 1)(1 + 2r/\epsilon)$  and  $\theta < \tau/d$ . First consider  $s = 1$ . Take  $r = r_0 \equiv \min\{k \in \mathbb{Z}^+ : k \geq 3, \frac{3}{2} \frac{1}{1-\beta}\}$  and  $\epsilon = 2$  in (4.26). Since  $\tau > d(2r - 1)(r + 1)$ , (4.26) yields,

$$\begin{aligned}
I_2 &\leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-2dr+d} \lambda_l^{2dr} \max_{0 \leq i \leq m_n - 1} \max_{\mathbf{k} \in \mathcal{K}_n} \left( \mathbb{E} |\tilde{X}_{\mathbf{k}\mathbf{0}}(\psi_i)|^{4(r+1)} \right)^{r/(r+1)} \\
&\quad + \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-dr} \lambda_l^{2dr} \max_{0 \leq i \leq m_n - 1} \max_{\mathbf{k} \in \mathcal{K}_n} \left( \mathbb{E} |\tilde{X}_{\mathbf{k}\mathbf{0}}(\psi_i)|^8 \right)^{r/2} \\
&\equiv I_{21} + I_{22}, \quad \text{say.}
\end{aligned} \tag{4.27}$$

Consider  $I_{22}$ . Since  $\tau > 35d$ , there exists an  $\epsilon_1 \in (0, 2)$  such that,

$$\tau > 7d(1 + 8/\epsilon_1) > 35d, \tag{4.28}$$

$$1 < \min\{4/(2 + \epsilon_1), 1 + 4/(8 + \epsilon_1) - 2/5\}. \tag{4.29}$$

Let  $\zeta > 1$  be such that  $\zeta$  is strictly less than the right side of (4.29). Then, applying Theorem 1.4.1.1 of Doukhan (1994), and using the fact that  $\mathbb{E} |\tilde{Y}_i(\mathbf{j})|^a \leq h_n^{-d} |\Delta u_i|$  for any  $a \geq 1$  and for all  $i \in \{0, 1, \dots, m_n - 1\}$ ,  $\mathbf{j} \in \mathcal{U}_{\mathbf{k}}$ , we have,

$$\begin{aligned}
I_{22} &\leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-dr} \lambda_l^{2dr} \\
&\quad \times \max_{0 \leq i \leq m_n - 1} \max_{\mathbf{k} \in \mathcal{K}_n} \left[ |\mathcal{B}_{\mathbf{0}}|^{-8} \left\{ \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_{8+\epsilon_1}^8 + \left( \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_{2+\epsilon_1}^2 \right)^4 \right\} \right]^{r/2} \\
&\leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-dr} \lambda_l^{2dr} \lambda_l^{-4dr} \max_{0 \leq i \leq m_n - 1} \left\{ \lambda_l^{dr/2} |\Delta u_i|^{4r/(8+\epsilon_1)} + \lambda_l^{2dr} |\Delta u_i|^{4r/(2+\epsilon_1)} \right\} \\
&\leq \text{const.} \lambda_n^{-dr} \lambda_n^{dr(\zeta/2 - 2/(8+\epsilon_1))} + \text{const.} (\lambda_n / \lambda_l)^{-dr} \lambda_n^{dr(\zeta/2 - 2/(2+\epsilon_1))} \\
&\leq \text{const.} \lambda_n^{-3d/2} \lambda_n^{-\gamma},
\end{aligned} \tag{4.30}$$

for some  $\gamma > 0$ , by our choice of  $\epsilon_1$  and  $\zeta$ . Indeed, for the first term,

$$\begin{aligned}
-dr + dr[\zeta/2 - 2/(8 + \epsilon_1)] &= -dr + dr(1/2 - 2/10) + dr[\zeta/2 - 2/(8 + \epsilon_1) - 1/2 + 2/10] \\
&= -7dr/10 + dr[\zeta - (1 + 4/(8 + \epsilon_1) - 2/5)]/2 < -2d.
\end{aligned}$$

And for the second term,  $(\lambda_n / \lambda_l)^{-dr} \leq \lambda_n^{-(1-\beta)dr} \leq \lambda_n^{-3d/2}$ , by the definition of  $r$ . To estimate  $I_{21}$ , again applying Theorem 1.4.1.1 of Doukan (1994), we have (with  $r = r_0$ ),

$$\begin{aligned}
I_{21} &\leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-2dr+d} \lambda_l^{2dr} \max_{0 \leq i \leq m_n - 1} \max_{\mathbf{k} \in \mathcal{K}_n} \left[ |\mathcal{B}_{\mathbf{0}}|^{-4(r+1)} \mathbb{E} \left( \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \tilde{Y}_i(\mathbf{j}) \right)^{4(r+1)} \right]^{r/(r+1)} \\
&\leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n / \lambda_l)^{-2dr+d} \lambda_l^{2dr} \times \max_{0 \leq i \leq m_n - 1} \max_{\mathbf{k} \in \mathcal{K}_n}
\end{aligned}$$

$$\begin{aligned}
& \left[ |\mathcal{B}_0|^{-4(r+1)} \left\{ \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_{4(r+1)+\epsilon_2}^{4(r+1)} + \left( \sum_{\mathbf{j} \in \mathcal{U}_{\mathbf{k}}} \|\tilde{Y}_i(\mathbf{j})\|_{2+\epsilon_2}^2 \right)^{2(r+1)} \right\} \right]^{r/(r+1)} \\
& \leq \text{const.} \lambda_n^{\zeta dr/2} (\lambda_n/\lambda_l)^{-2dr+d} \lambda_l^{2dr} \lambda_l^{-4dr} \\
& \quad \times \max_{0 \leq i \leq m_n-1} \left\{ \lambda_l^{dr/(r+1)} |\Delta u_i|^{4r/(4(r+1)+\epsilon_2)} + \lambda_l^{2dr} |\Delta u_i|^{4r/(2+\epsilon_2)} \right\} \\
& \leq \text{const.} \lambda_n^{\zeta dr/2} \lambda_n^{-5d} + \text{const.} \lambda_n^{\zeta dr/2} \lambda_n^{-(1-\beta)(2dr-d)} \lambda_n^{-2dr/(2+\epsilon_2)}, \tag{4.31}
\end{aligned}$$

for some  $\epsilon_2 > 0$ , provided  $\sum_{i=1}^{\infty} i^{d(4(r+1)-1)-1} i^{-\tau\epsilon_2/(4(r+1)+\epsilon_2)} < \infty$ , that is,

$$\tau > d(4r+3)[1+4(r+1)/\epsilon_2]. \tag{4.32}$$

Since  $\tau > d(4r+3)$ , there exists  $\epsilon_2 > 0$  arbitrarily large, such that (4.32) holds. When  $0 < \beta \leq 1/4$ ,

$$\begin{aligned}
\zeta r/2 - (1-\beta)(2r-1) & \leq \zeta r/2 - 3(r-1)/4 - (1-\beta)r \\
& = r(\zeta-1)/2 - r/4 + 3/4 - (1-\beta)r \\
& \leq r(\zeta-1)/2 - 3/2.
\end{aligned}$$

Now choose  $\zeta > 1$  such that  $r(\zeta-1)/2 - 3/2 - 2r/(2+\epsilon_2) < -3/2$ , that is,  $1 < \zeta < 1 + 4/(2+\epsilon_2)$ . Then, with this choice of  $\epsilon_2$  and  $\zeta$ , the last line of (4.31) is summable and therefore, (4.21) holds. When  $1/4 < \beta \leq d/(d+2)$ , choose  $\epsilon_2 \in (0, (14-12\beta)/(4\beta-1))$  such that,  $\tau > d(4r+3)[1+4(r+1)/\epsilon_2] > d(4r+3)[1+4(r+1)(4\beta-1)/(14-12\beta)]$ . Then, it can be shown as in the  $\beta \in (0, 1/4]$  case that there exists a  $\zeta > 1$  such that the exponent of  $\lambda_n^d$  in the second term of (4.31) is less than  $-3/2 - \gamma$  for some  $\gamma > 0$ . Therefore, (4.21) holds.

The proof of (4.22) is similar to that of (4.21). Here, we set  $r=3, s=2, \epsilon=2$  in (4.26) and repeatedly apply Theorem 1.4.1.1 of Doukhan (1994) in carrying out steps analogous to (4.27), (4.30), and (4.31) (where we set both  $\epsilon_1 = \epsilon_2 = (14-12\beta)/[3(2\beta-1)]$  for  $\beta > 1/2$  and  $\epsilon_1, \epsilon_2 \in (0, \infty)$  arbitrarily large for  $0 < \beta \leq 1/2$ ). We omit the details to save space.

**Lemma 4.9** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > 77d, \theta < \tau/d$ . Then, there exists a set  $A \in \mathcal{F}$  with  $\mathbb{P}(A) = 1$  such that for any  $\omega \in A$ , for any given  $u_0 \in [0, 1]$ , for every  $\epsilon > 0$ , there exist a  $\delta \in (0, 1)$  and an  $N(\omega) \in \mathbb{Z}^+$ , such that for all  $n \geq N(\omega)$ ,*

$$\max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i))| \leq \epsilon/2.$$

**Proof:** Note that,

$$\begin{aligned}
\max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i))| & \leq \max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i| + \max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} \Delta u_i \\
& \leq \max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i| + \epsilon/6.
\end{aligned}$$

Hence, it suffices to show that  $\max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i| \rightarrow 0$  a.s, which holds if the upper bound of,

$$\mathbb{P}\left(\max_{0 \leq i \leq m_n-1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i| \geq \gamma\right) \leq \sum_{0 \leq i \leq m_n-1} \gamma^{-2r} \mathbb{E}[\lambda_n^{d/2} (\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i)]^{2r}, \tag{4.33}$$

is summable over  $n$ , for a properly chosen  $r \in \mathbb{Z}^+$ .

Rewrite,

$$\begin{aligned}
\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i &= \left( \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{\mathbf{k}}| \right)^{-1} |\mathcal{B}_{\mathbf{0}}| \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_*(\tilde{X}_{\mathbf{k}}^*(\psi_i)) \\
&= \left( \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{B}_{\mathbf{k}}| \right)^{-1} |\mathcal{B}_{\mathbf{0}}| \sum_{\mathbf{k} \in \mathcal{K}_n} |\mathcal{T}_n|^{-1} \sum_{\mathbf{t} \in \mathcal{T}_n} \tilde{X}_{\mathbf{k}\mathbf{t}}(\psi_i) \\
&= |\mathcal{T}_n|^{-1} \sum_{\mathbf{s} \in \mathcal{S}_n} \tilde{\omega}_n(\mathbf{s}) [I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i],
\end{aligned}$$

where the constants  $\tilde{\omega}_n(\cdot) \in [0, 2]$ . Hence,

$$\begin{aligned}
\mathbb{E}[\mathbb{E}_*(\tilde{F}_n^*(\psi_i)) - \Delta u_i]^{2r} &= |\mathcal{T}_n|^{-2r} \mathbb{E} \left[ \sum_{\mathbf{s} \in \mathcal{S}_n} \tilde{\omega}_n(\mathbf{s}) (I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i) \right]^{2r} \\
&= |\mathcal{T}_n|^{-2r} \mathbb{E} \left[ \sum_{\mathbf{l} \in \mathcal{L}_n} \tilde{W}(\mathbf{l}) \right]^{2r},
\end{aligned}$$

where  $\tilde{W}(\mathbf{l}) \equiv \sum_{\mathbf{s} \in (\mathbf{l} + [0, 1]^d) \cap \mathcal{S}_n} \tilde{\omega}_n(\mathbf{s}) [I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i]$  denotes the sum over the unit block  $(\mathbf{l} + [0, 1]^d) \cap \mathcal{S}_n$ ;  $\mathbf{l} \in \mathcal{L}_n$ . Note that for a given  $a > 0$ ,

$$\begin{aligned}
\mathbb{E}|\tilde{W}(\mathbf{l})|^a &\leq \mathbb{E} \left[ \sum_{\mathbf{s} \in (\mathbf{l} + [0, 1]^d) \cap \mathcal{S}_n} \tilde{\omega}_n(\mathbf{s}) (I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i) \right]^a \\
&\leq \text{const.} (h_n^{-d})^a \mathbb{E} |I(G^{-1}(u_i) < Z(\mathbf{s}) \leq G^{-1}(u_{i+1})) - \Delta u_i|^a \\
&\leq \text{const.} (h_n^{-d})^a \Delta u_i.
\end{aligned}$$

Take  $r = 6$ . Since  $\tau > 77d$ , there exists  $\epsilon \in (0, 2)$  such that  $\tau > d(2r - 1)(1 + 2r/\epsilon) > 11d(1 + 12/2)$ . Hence, with this  $\epsilon$ , by Theorem 1.4.1.1 of Doukhan (1994),

$$\begin{aligned}
|\mathcal{T}_n|^{-2r} \mathbb{E} \left[ \sum_{\mathbf{l} \in \mathcal{L}_n} \tilde{W}(\mathbf{l}) \right]^{2r} &\leq \text{const.} |\mathcal{T}_n|^{-2r} \max \left\{ \sum_{\mathbf{l} \in \mathcal{L}_n} \|\tilde{W}(\mathbf{l})\|_{2r+\epsilon}^{2r}, \left[ \sum_{\mathbf{l} \in \mathcal{L}_n} \|\tilde{W}(\mathbf{l})\|_{2+\epsilon}^2 \right]^r \right\} \\
&\leq \text{const.} (\lambda_n^d h_n^{-d})^{12} \max \left\{ \lambda_n^d h_n^{-2dr} |\Delta u_i|^{12/(12+\epsilon)}, (\lambda_n^d h_n^{-2d} |\Delta u_i|^{2/(2+\epsilon)})^6 \right\} \\
&\leq \text{const.} \lambda_n^{-15d/2-\gamma},
\end{aligned}$$

for some  $\gamma > 0$ . This shows that the right side of (4.33) is summable over  $n$ . Hence the result of the lemma holds.  $\square$

**Lemma 4.10** *Under the asymptotic framework of Section 2.2 and the bootstrap procedure given in Section 2.3, suppose that assumptions (A.1)–(A.5) hold, and that the mixing coefficient parameters  $\tau$  and  $\theta$  satisfy  $\tau > \max\{77d, \tau_1^0, \tau_2^0, \tau_3^0\}$ ,  $\theta < \tau/d$ . Then, there exists a set  $A \in \mathcal{F}$  with  $\mathbb{P}(A) = 1$  such that for all  $\omega \in A$ , for any given  $u_0 \in [0, 1]$ , for every  $\epsilon > 0$ ,  $\eta > 0$ , there exist a  $\delta \in (0, 1)$  and an  $N(\omega) \in \mathbb{Z}^+$ , such that for all  $n \geq N(\omega)$ ,*

$$\mathbb{P}_* \left( \sup_{u_0 \leq u \leq u_0 + \delta} |\tilde{\xi}_n^*(u) - \tilde{\xi}_n^*(u_0)| \geq \epsilon \right) \leq \eta \delta.$$

**Proof:** By (22.18) of Billingsley (1968),

$$\begin{aligned} \sup_{u_0 \leq u \leq u_0 + \delta} |\tilde{\xi}_n^*(u) - \tilde{\xi}_n^*(u_0)| &\leq 3 \max_{0 \leq i \leq m_n - 1} |\tilde{\xi}_n^*(u_{i+1}) - \tilde{\xi}_n^*(u_i)| \\ &\quad + \max_{0 \leq i \leq m_n - 1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(u_{i+1})) - \mathbb{E}_*(\tilde{F}_n^*(u_i))| \end{aligned}$$

By Lemma 4.8 and Lemma 4.9, there exist a  $\delta > 0$  and an  $N(\omega) \in \mathbb{Z}^+$  such that, for all  $n \geq N(\omega)$ ,

$$\mathbb{P}_* \left( \max_{0 \leq i \leq m_n - 1} |\tilde{\xi}_n^*(u_{i+1}) - \tilde{\xi}_n^*(u_i)| \geq \epsilon/6 \right) \leq \eta\delta/2,$$

and  $\max_{0 \leq i \leq m_n - 1} \lambda_n^{d/2} |\mathbb{E}_*(\tilde{F}_n^*(u_{i+1})) - \mathbb{E}_*(\tilde{F}_n^*(u_i))| \leq \epsilon/2$  for all  $\omega \in A$ . Hence the result of the lemma holds.  $\square$

### Proof of Theorem 3.2

Since the marginal CDF  $G(\cdot)$  is continuous by assumption (A.1), it suffices to show that, the rescaled process converges weakly a.s.; that is,

$$\tilde{\xi}_n^* \xrightarrow{\mathcal{D}} \tilde{\mathcal{W}} \text{ a.s.}, \quad (4.34)$$

where  $\xrightarrow{\mathcal{D}}$  denotes weak convergence of random elements in  $D[0, 1]$ ,  $\tilde{\xi}_n^*(\cdot) = \xi_n^*(G^{-1}(\cdot))$  and  $\tilde{\mathcal{W}}(\cdot) = \mathcal{W}(G^{-1}(\cdot))$  are defined as before.

By Lemma 4.7, the finite dimensional distributions of  $\tilde{\xi}_n^*(\cdot)$  converge weakly to those of  $\tilde{\mathcal{W}}(\cdot)$  a.s. By Lemma 4.10,  $\tilde{\mathcal{W}}(\cdot)$  is tight. Hence, by Theorem 15.1 in Billingsley (1968), the result of (4.34) and hence, Theorem 3.2 holds.  $\square$

### 4.3 Proof of Theorem 3.3

As in the univariate case, the proof consists of two steps, namely, (I) showing the weak convergence of finite dimensional distributions and (II) proving the tightness of the relevant process. For  $\xi_n^{(m)}$ , step (I) is a simple consequence of the Cramér-Wold device and the CLT result of Lahiri (2000) (c.f. Theorem 4.1). For  $\xi_n^{(m)*}$ , step (I) easily follows by repeating the steps in the proof of Section 4.2. Thus, it remains to establish step (II). Using the arguments developed by Bühlmann (1994), it is enough to show that for any given  $\mathbf{u}_0 \in [0, 1]^m$ , for every  $\epsilon > 0$ ,  $\eta > 0$ , there exist  $\delta \in (0, 1)$  and  $n_0 \geq 1$  such that for all  $n \geq n_0$ ,

$$\mathbb{P} \left( \sup_{\mathbf{u} \in Q(\mathbf{u}_0, \delta)} |\tilde{\xi}_n^{(m)}(\mathbf{u}) - \tilde{\xi}_n^{(m)}(\mathbf{u}_0)| \geq \epsilon \right) \leq \eta\delta, \quad (4.35)$$

and

$$\mathbb{P}_* \left( \sup_{\mathbf{u} \in Q(\mathbf{u}_0, \delta)} |\tilde{\xi}_n^{(m)*}(\mathbf{u}) - \tilde{\xi}_n^{(m)*}(\mathbf{u}_0)| \geq \epsilon \right) \leq \eta\delta \text{ a.s.}, \quad (4.36)$$

where  $Q(\mathbf{u}_0, \delta) \equiv \{\mathbf{u} \in [0, 1]^m : \mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{u}_0 + \delta \mathbf{1}\}$ ,  $\mathbf{1} = (1, \dots, 1)' \in \mathbb{R}^m$ , and  $\tilde{\xi}_n^{(m)}$  and  $\tilde{\xi}_n^{(m)*}$  are the rescaled versions of  $\xi_n^{(m)}$  and  $\xi_n^{(m)*}$ , defined by  $\tilde{\xi}_n^{(m)}(\mathbf{u}) \equiv \xi_n^{(m)}(G_1^{-1}(u_1), \dots, G_m^{-1}(u_m))$ , and  $\tilde{\xi}_n^{(m)*}(\mathbf{u}) \equiv \xi_n^{(m)*}(G_1^{-1}(u_1), \dots, G_m^{-1}(u_m))$ ,  $\mathbf{u} = (u_1, \dots, u_m)' \in [0, 1]^m$ . Here,  $G_i(z) \equiv \mathbb{P}(Z_i(\mathbf{0}) \leq z)$ ,  $z \in \mathbb{R}$  denotes the marginal distribution function of  $Z_i(\mathbf{0})$ ,  $1 \leq i \leq m$ .

First, consider (4.36). A proof may be constructed by using the error bounds derived above for the univariate r.f. case while retracing the steps of Bühlmann (1994, pp.1003–1006). In particular, using arguments similar to the proof of (4.17) above and using Theorem 1.4.1.1 of Doukhan (1994) in place of the moment bounds of Yokoyama (1980), one may derive the following analog of Bühlmann (1994)'s key inequality (15) (cf. pp. 1003) for the spatial bootstrap case.

For a bounded measurable function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  with  $\mathbb{E}(f(\mathbf{Z}(\mathbf{0}))) = 0$ , and any integer  $r \geq 1$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sum_{\mathbf{k} \in \mathcal{K}_n} \mathbb{E}_* |(\lambda_l^d h_n^{2d})^{-1/2} \sum_{\mathbf{s} \in \mathcal{B}_{\mathbf{k}}^*} f(\mathbf{Z}(\mathbf{s}))|^{2r} \right] \\ & \leq C(d, r, m) |\mathcal{K}_n| \max\{(\mathbb{E}|f(\mathbf{Z}(\mathbf{0}))|^{12r})^{1/6}, (\mathbb{E}|f(\mathbf{Z}(\mathbf{0}))|^2)^r\} (\lambda_l/\lambda_n)^{dr}, \end{aligned}$$

provided  $\tau > 3d(4r - 1)/2$  and  $\theta < \tau/d$ .

Similarly, one needs to use arguments similar to the proof of Lemma 4.9 above to establish an analog of Bühlmann (1994)'s Lemma 10, which in turn allows one to prove (4.36). The proof of (4.35) is similar to (and indeed simpler than) the proof of (4.36). We omit the details.

#### 4.4 Proof of Theorem 3.4

By the Fréchet differentiability of  $\mathbf{T}$  at  $G^{(m)}$ , we have,

$$\mathbf{T}(G_n^{(m)}) - \mathbf{T}(G^{(m)}) = \int \mathbf{g} d(G_n^{(m)} - G^{(m)}) + \mathbf{R}(G_n^{(m)}), \quad (4.37)$$

where for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that,

$$\|\mathbf{R}(G_n^{(m)})\| \leq \epsilon \|G_n^{(m)} - G^{(m)}\|_\infty, \quad (4.38)$$

whenever  $\|G_n^{(m)} - G^{(m)}\|_\infty \leq \delta$ . Part (i) of Theorem 3.4 now follows from Theorem 3.3 (i), (4.37), (4.38), Slutsky's Theorem, and Theorem 4.1 of Lahiri (2000).

Next, consider part (ii). Without loss of generality, suppose that  $\int \mathbf{g} dG^{(m)} = \mathbf{0}$ . Then, applying (4.37) twice, we have,

$$\lambda_n^{d/2}(\boldsymbol{\theta}_n^* - \tilde{\boldsymbol{\theta}}_n) = \lambda_n^{d/2}(\bar{\mathbf{g}}_n^* - \tilde{\mathbf{g}}_n) + \mathbf{R}_n^*, \quad (4.39)$$

where  $\bar{\mathbf{g}}_n^* \equiv \int \mathbf{g} dF_n^{(m)*}$ ,  $\tilde{\mathbf{g}}_n \equiv \int \mathbf{g} dE_*(F_n^{(m)*})$ , and  $\mathbf{R}_n^* \equiv \lambda_n^{d/2}[\mathbf{R}(F_n^{(m)*}) + \mathbf{R}(E_*(F_n^{(m)*}))]$ . Since  $G_1, \dots, G_m$  are continuous on  $\mathbb{R}$ , by arguments similar to the proof of Lemma 4.9,

$$\lambda_n^{d/2} \|E_*(F_n^{(m)*}) - G^{(m)}\|_\infty = \mathcal{O}_p(1). \quad (4.40)$$

Also, by (4.40) and Theorem 3.3 (ii), the conditional distribution of  $\lambda_n^{d/2} \|F_n^{(m)*} - G^{(m)}\|_\infty$  is a tight sequence, a.s. Hence, by (4.38), for any  $\epsilon > 0$ ,

$$\mathbb{P}_*(\|\mathbf{R}_n^*\| > \epsilon) = o_p(1). \quad (4.41)$$

Also, using arguments similar to those in Section 4.2, we have,

$$\lambda_n^{d/2}(\bar{\mathbf{g}}_n^* - \tilde{\mathbf{g}}_n) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \Sigma_0) \text{ a.s.} \quad (4.42)$$

Hence, by Corollary 2.6 and Corollary 3.2 of Bhattacharya and Rao (1986), Theorem 3.4 (ii) follows from (4.39), (4.41), and (4.42).

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