Example 7.27 (Multiple comparison in one-way ANOVA models)

Consider the one-way ANOVA model

\[ X_{ij} = N(\mu_i, \sigma^2), \quad j = 1, \ldots, n_i, i = 1, \ldots, m. \]

If the hypothesis \( H_0 : \mu_1 = \cdots = \mu_m \) is rejected, one typically would like to compare \( \mu_i \)'s. One way to compare \( \mu_i \)'s is to consider simultaneous confidence intervals for \( \mu_i - \mu_j, 1 \leq i < j \leq m \).

Since \( X_{ij} \)'s are independently normal, the sample means \( \bar{X}_i \) are independently normal \( N(\mu_i, \sigma^2/n_i), i = 1, \ldots, m \), respectively, and they are independent of

\[ SSR = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2. \]

Consequently, \( (\bar{X}_i - \bar{X}_j)/\sqrt{v_{ij}} \) has the t-distribution \( t_{n-m}, 1 \leq i < j \leq m \), where \( v_{ij} = (n_i^{-1} + n_j^{-1})SSR/(n - m) \).
Example 7.27 (continued)

For each \((i, j)\), a confidence interval for \(\mu_i - \mu_j\) with confidence coefficient \(1 - \alpha\) is

\[
C_{ij,\alpha}(X) = [\bar{X}_i - \bar{X}_j - t_{n-m,\alpha/2} \sqrt{v_{ij}}, \bar{X}_i - \bar{X}_j + t_{n-m,\alpha/2} \sqrt{v_{ij}}],
\]

where \(t_{n-m,\alpha}\) is the \((1 - \alpha)\)th quantile of the t-distribution \(t_{n-m}\).

One can show that \(C_{ij,\alpha}(X)\) is actually UMAU (exercise).

Bonferroni’s level \(1 - \alpha\) simultaneous confidence intervals for \(\mu_i - \mu_j\), \(1 \leq i < j \leq m\), are \(C_{ij,\alpha_{\star}}(X)\), \(1 \leq i < j \leq m\), where \(\alpha_{\star} = 2\alpha / [m(m-1)]\).

When \(m\) is large, these confidence intervals are very conservative in the sense that the confidence coefficient of these intervals may be much larger than the nominal level \(1 - \alpha\) and these intervals may be too wide to be useful.

If the normality assumption is removed, then \(C_{ij,\alpha}(X)\) is \(1 - \alpha\) asymptotically correct as \(\min\{n_1, \ldots, n_m\} \to \infty\) and \(\max\{n_1, \ldots, n_m\} / \min\{n_1, \ldots, n_m\} \to c < \infty\).

Therefore, \(C_{ij,\alpha_{\star}}(X)\), \(1 \leq i < j \leq m\), are simultaneous confidence intervals with asymptotic confidence level \(1 - \alpha\).
Example 7.27 (continued)

For each \( (i,j) \), a confidence interval for \( \mu_i - \mu_j \) with confidence coefficient \( 1 - \alpha \) is

\[
C_{ij,\alpha}(X) = [\bar{X}_i - \bar{X}_j - t_{n-m,\alpha/2} \sqrt{\bar{v}_{ij}}, \bar{X}_i - \bar{X}_j + t_{n-m,\alpha/2} \sqrt{\bar{v}_{ij}}],
\]

where \( t_{n-m,\alpha} \) is the \( (1 - \alpha) \)th quantile of the t-distribution \( t_{n-m} \).

One can show that \( C_{ij,\alpha}(X) \) is actually UMAU (exercise).

Bonferroni’s level \( 1 - \alpha \) simultaneous confidence intervals for \( \mu_i - \mu_j \), \( 1 \leq i < j \leq m \), are \( C_{ij,\alpha_*}(X), 1 \leq i < j \leq m \), where \( \alpha_* = 2\alpha/[m(m-1)] \).

When \( m \) is large, these confidence intervals are very conservative in the sense that the confidence coefficient of these intervals may be much larger than the nominal level \( 1 - \alpha \) and these intervals may be too wide to be useful.

If the normality assumption is removed, then \( C_{ij,\alpha}(X) \) is \( 1 - \alpha \) asymptotically correct as \( \min\{n_1, \ldots, n_m\} \to \infty \) and \( \max\{n_1, \ldots, n_m\}/\min\{n_1, \ldots, n_m\} \to c < \infty \).

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Example 7.27 (continued)

For each \((i, j)\), a confidence interval for \(\mu_i - \mu_j\) with confidence coefficient \(1 - \alpha\) is

\[
C_{ij,\alpha}(X) = [\bar{X}_i - \bar{X}_j - t_{n-m,\alpha/2} \sqrt{v_{ij}}, \bar{X}_i - \bar{X}_j + t_{n-m,\alpha/2} \sqrt{v_{ij}}],
\]

where \(t_{n-m,\alpha}\) is the \((1 - \alpha)\)th quantile of the t-distribution \(t_{n-m}\).

One can show that \(C_{ij,\alpha}(X)\) is actually UMAU (exercise).

Bonferroni’s level \(1 - \alpha\) simultaneous confidence intervals for \(\mu_i - \mu_j\), \(1 \leq i < j \leq m\), are \(C_{ij,\alpha_\star}(X)\), \(1 \leq i < j \leq m\), where \(\alpha_\star = 2\alpha/[m(m-1)]\).

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Therefore, \(C_{ij,\alpha_\star}(X)\), \(1 \leq i < j \leq m\), are simultaneous confidence intervals with asymptotic confidence level \(1 - \alpha\).
We can also use Scheffé’s method to obtain simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \leq i < j \leq m$.

Scheffé’s intervals have confidence coefficient $1 - \alpha$ for $t^\tau L \beta$, $t \in \mathcal{T}$, where $\beta = (\mu_1, \ldots, \mu_m)$ and

$$L = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & 0 & \cdots & 0 & -1 \\
0 & 0 & 1 & \cdots & 0 & -1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}.$$  

Scheffé’s intervals are also too conservative.

In fact, they are often more conservative than Bonferroni’s intervals.

Tukey’s method in one-way ANOVA models

Consider the one-way ANOVA model in Example 7.27. Tukey’s method introduced next produces simultaneous confidence intervals for all nonzero contrasts (including the differences $\mu_i - \mu_j$, $1 \leq i < j \leq m$) with confidence coefficient $1 - \alpha$. 
Example 7.27 (continued)

We can also use Scheffé’s method to obtain simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \leq i < j \leq m$.
Scheffé’s intervals have confidence coefficient $1 - \alpha$ for $t^\tau L \beta$, $t \in \mathcal{T}$, where $\beta = (\mu_1, ..., \mu_m)$ and

$$L = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -1 \\
0 & 1 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}.$$

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Studentized range

Let \( \hat{\sigma}^2 = \frac{SSR}{(n - m)} \), where SSR is given in Example 7.27. The studentized range is defined to be

\[
R_{st} = \max_{1 \leq i < j \leq m} \frac{|(\bar{X}_i - \mu_i) - (\bar{X}_j - \mu_j)|}{\hat{\sigma}}.
\]

The distribution of \( R_{st} \) does not depend on any unknown parameter.

Theorem 7.11

Assume the one-way ANOVA model in Example 7.27. Let \( q_\alpha \) be the \((1 - \alpha)\)th quantile of the studentized range \( R_{st} \). Then Tukey’s intervals

\[
[c^\tau \hat{\beta} - q_\alpha \hat{\sigma} c_+, c^\tau \hat{\beta} + q_\alpha \hat{\sigma} c_+] , \quad c \in \mathbb{R}^m - \{0\}, c^\tau J = 0,
\]

are simultaneous confidence intervals for \( c^\tau \beta \), \( c \in \mathbb{R}^m - \{0\}, c^\tau J = 0 \), with confidence coefficient \( 1 - \alpha \), where \( c_+ \) is the sum of all positive components of \( c \), \( \beta = (\mu_1, ..., \mu_m) \), \( \hat{\beta} = (\bar{X}_1, ..., \bar{X}_m) \), and \( J \) is the \( m \)-vector of ones.
Studentized range

Let $\hat{\sigma}^2 = SSR/(n - m)$, where $SSR$ is given in Example 7.27. The *studentized range* is defined to be

$$R_{st} = \max_{1 \leq i < j \leq m} \frac{|(\bar{X}_i - \mu_i) - (\bar{X}_j - \mu_j)|}{\hat{\sigma}}.$$ 

The distribution of $R_{st}$ does not depend on any unknown parameter.

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$$[c^\tau \hat{\beta} - q_\alpha \hat{\sigma} c_+, c^\tau \hat{\beta} + q_\alpha \hat{\sigma} c_+], \quad c \in \mathbb{R}^m - \{0\}, c^\tau J = 0,$$

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Proof

Let \( Y_i = (\bar{X}_i - \mu_i)/\hat{\sigma} \) and \( Y = (Y_1, \ldots, Y_m) \).
Then the result follows if we can show that

\[
\max_{1 \leq i < j \leq m} |Y_i - Y_j| \leq q_\alpha
\]

is equivalent to

\[
|c^\tau Y| \leq q_\alpha c_+ \quad \text{for all } c \in \mathbb{R}^m \text{ satisfying } c^\tau J = 0, c \neq 0.
\]

Let \( c(i,j) = (c_1, \ldots, c_m) \) with \( c_i = 1, \quad c_j = -1, \quad \text{and } c_l = 0 \text{ for } l \neq i \text{ or } l \neq j. \)
Then

\[
c(i,j)_+ = 1 \quad \text{and} \quad |[c(i,j)]^\tau Y| = |Y_i - Y_j|.
\]

Therefore, (2) implies (1).

Next, we show (1) implies (2).
Let \( c = (c_1, \ldots, c_m) \) be a vector satisfying the conditions in (2).
Define \(-c_-\) to be the sum of negative components of \( c \).
Then the result follows from
Proof

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Then the result follows if we can show that

\[
\max_{1 \leq i < j \leq m} |Y_i - Y_j| \leq q_\alpha \tag{1}
\]

is equivalent to

\[
|c^\tau Y| \leq q_\alpha c_+ \quad \text{for all } c \in \mathbb{R}^m \text{ satisfying } c^\tau J = 0, c \neq 0. \tag{2}
\]

Let \( c(i, j) = (c_1, \ldots, c_m) \) with \( c_i = 1, c_j = -1, \) and \( c_l = 0 \) for \( l \neq i \) or \( l \neq j \).
Then

\[
c(i, j)_+ = 1 \quad \text{and} \quad |[c(i, j)]^\tau Y| = |Y_i - Y_j|.
\]

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Define \(-c_-\) to be the sum of negative components of \( c \).
Then the result follows from
Proof (continued)

\[ |c^\tau Y| = \frac{1}{c_+} \left| c_+ \sum_{j:c_j<0} c_j Y_j + c_- \sum_{i:c_i>0} c_i Y_i \right| \]

\[ = \frac{1}{c_+} \left| \sum_{i:c_i>0} \sum_{j:c_j<0} c_i c_j Y_j - \sum_{j:c_j<0} \sum_{i:c_i>0} c_i c_j Y_i \right| \]

\[ = \frac{1}{c_+} \left| \sum_{i:c_i>0} \sum_{j:c_j<0} c_i c_j (Y_j - Y_i) \right| \]

\[ \leq \frac{1}{c_+} \sum_{i:c_i>0} \sum_{j:c_j<0} |c_i c_j| |Y_j - Y_i| \]

\[ \leq \max_{1 \leq i < j \leq m} |Y_j - Y_i| \left( \frac{1}{c_+} \sum_{i:c_i>0} \sum_{j:c_j<0} |c_i||c_j| \right) \]

\[ = \max_{1 \leq i < j \leq m} |Y_j - Y_i| c_+, \]

where the first and the last equalities follow from the fact that \( c_- = c_+ \neq 0 \).
Remarks

- Tukey’s method works well when \( n_i \)'s are all equal to \( n_0 \), in which case values of \( \sqrt{n_0} q_\alpha \) can be found using tables or statistical software.

- When \( n_i \)'s are unequal, some modifications are suggested; see Tukey (1977) and Milliken and Johnson (1992).

Example 7.29

We compare the t-type confidence intervals, Bonferroni’s, Scheffé’s, and Tukey’s simultaneous confidence intervals for \( \mu_i - \mu_j \), \( 1 \leq i < j \leq 3 \), based on the following data \( X_{ij} \) given in Mendenhall and Sincich (1995):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>148</td>
<td>76</td>
<td>393</td>
<td>520</td>
<td>236</td>
<td>134</td>
<td>55</td>
<td>166</td>
<td>415</td>
<td>153</td>
</tr>
<tr>
<td>2</td>
<td>513</td>
<td>264</td>
<td>433</td>
<td>94</td>
<td>535</td>
<td>327</td>
<td>214</td>
<td>135</td>
<td>280</td>
<td>304</td>
</tr>
<tr>
<td>3</td>
<td>335</td>
<td>643</td>
<td>216</td>
<td>536</td>
<td>128</td>
<td>723</td>
<td>258</td>
<td>380</td>
<td>594</td>
<td>465</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>i = 1</strong></td>
<td>148</td>
<td>76</td>
<td>393</td>
<td>520</td>
<td>236</td>
<td>134</td>
<td>55</td>
<td>166</td>
<td>415</td>
<td>153</td>
</tr>
<tr>
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<td>723</td>
<td>258</td>
<td>380</td>
<td>594</td>
<td>465</td>
</tr>
</tbody>
</table>
Example 7.29 (continued)

In this example, $m = 3$, $n_i = n_0 = 10$, $\bar{X}_1 = 229.6$, $\bar{X}_2 = 309.8$, $\bar{X}_3 = 427.8$, and $\hat{\sigma} = 168.95$.

Let $\alpha = 0.05$.

For the t-type intervals, $t_{27,0.975} = 2.05$.

For Bonferroni’s method, $\alpha_\ast = \alpha / 3 = 0.017$ and $t_{27,0.983} = 2.55$.

For Scheffé’s method, $c_{0.05} = 3.35$ and $\sqrt{2c_{0.05}} = 2.59$.

From Table 13 in Mendenhall and Sincich (1995, Appendix II), $\sqrt{n_0 q_{0.05}} = 3.49$.

The resulting confidence intervals are given in the following table.

<table>
<thead>
<tr>
<th>Interval Type</th>
<th>$c_{0.05}$</th>
<th>$\sqrt{2c_{0.05}}$</th>
<th>$\sqrt{n_0 q_{0.05}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t-type</td>
<td>3.35</td>
<td>2.59</td>
<td>3.49</td>
</tr>
<tr>
<td>Bonferroni</td>
<td>0.017</td>
<td>2.55</td>
<td>3.49</td>
</tr>
<tr>
<td>Scheffé</td>
<td>3.35</td>
<td>2.59</td>
<td>3.49</td>
</tr>
</tbody>
</table>
**Example 7.29 (continued)**

<table>
<thead>
<tr>
<th>Method</th>
<th>( \mu_1 - \mu_2 )</th>
<th>( \mu_1 - \mu_3 )</th>
<th>( \mu_2 - \mu_3 )</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>t-type</td>
<td>([-235.2, 74.6])</td>
<td>([-353.1, -43.3])</td>
<td>([-272.8, 37.0])</td>
<td>309.8</td>
</tr>
<tr>
<td>Bonferroni</td>
<td>([-273.0, 112.4])</td>
<td>([-390.9, -5.5])</td>
<td>([-310.6, 74.8])</td>
<td>385.4</td>
</tr>
<tr>
<td>Scheffé</td>
<td>([-276.0, 115.4])</td>
<td>([-393.9, -2.5])</td>
<td>([-313.6, 77.8])</td>
<td>391.4</td>
</tr>
<tr>
<td>Tukey</td>
<td>([-267.3, 106.7])</td>
<td>([-385.2, -11.2])</td>
<td>([-304.9, 69.1])</td>
<td>374.0</td>
</tr>
</tbody>
</table>

**Discussions**

- Apparently, t-type intervals have the shortest length, but they are not simultaneous confidence intervals.
- Tukey’s intervals in this example have the shortest length among simultaneous confidence intervals.
- Scheffé’s intervals have the longest length.