Stat 992: Lecture 24
Maxima of random fields II.

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1. *Intrinsic volumes.*. Read Worsley, K.J. (2003). Detecting activation in fMRI data. Statistical Methods in Medical Research, 12:401-418 for a fast overview. Cao, J. and Worsley, K.J. (2001). Applications of random fields in human brain mapping. In M. Moore (Ed.) Spatial Statistics: Methodological Aspects and Applications, Springer Lecture Notes in Statistics, 159:169-182.

   For any smooth random fields Y, define excursion set $A_h = \{x : Y(x) > h\}$. Then it was shown by Adler that

$$P(\sup_{x\in\Omega} Y(x) > h) \approx \mathbb{E}\chi(\Omega\cup A_h) = \sum_{d=0}^{N} \mu_d(\Omega)\rho_d(h)$$

   where μ_d is d-dimensional Minkowski functional or *intrinsic volume* of Ω and ρ_d is d-dimentional Euler characteristic (EC) density of Y. Most inference in imaging is done in 3D so $N = 3$. For details on intrinsic volume, read V. Schmidt and E. Spodarev's Joint Estimators for the Specific Intrinsic Volumes of Stationary Random Sets (2003).

   For search region, we can always contain Ω in a sphere with radius r. Then the intrinsic volumes are $\mu_0 = \chi(\Omega) = 1$, $\mu_1 = 4r$, $\mu_2 = \|\partial\Omega\| = 4\pi r^2$, $\mu_3 = \|\Omega\| = \frac{4}{3}\pi r^3$. If we want to use a 3D rectangle of size $a \times b \times c$, the intrinsic volumes are $\mu_0 = 1, \mu_1 = a + b + c, \mu_2 = ab + bc + ac, \mu_3 = abc$. For regular image grid system with equal voxel size δ in all direction, the intrinsic volume can be computed in the following fashion. Let V be the total number of vertices that forms the corners of voxels, E be the total number of edges connecting each vertices, F be the number of faces formed by 4 connected edges. Then $\mu_0 = \chi(\Omega) = V - E + F, \mu_1 = (E - 2F)\delta, \mu_2 = F\delta^2$.

2. *Example: corpus callosum of the brain.* The corpus callosum (CC) Ω is the white matter region that connects the left hemisphere to the right hemisphere. The average shape of midsagittal cross-section is given in Figure 1. To find the number

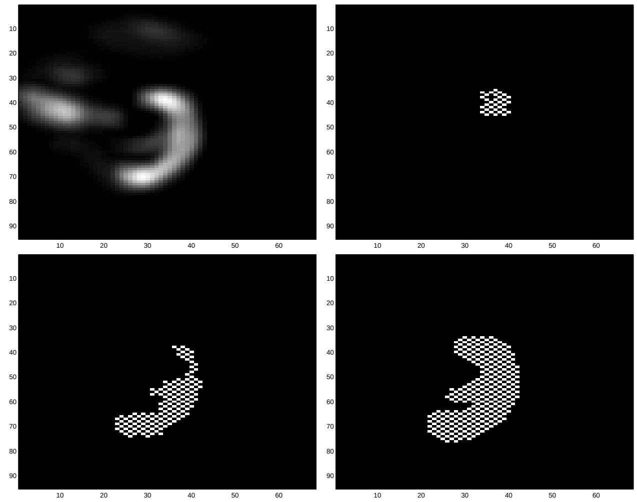

Figure 1: Top left: original white matter density of the corpus callosum. Search region Ω is defined as the white matter region with density higher than 0.1. It will be a little bit larger than the actual corpus callosum shape.

   of edges and pixels contained in Ω, we start from a seed pixel $p_0 = (40, 40)$ that is the splenium of CC and implement random search algorithm and found $V = 328$, $E = 599$ and $F = 272$. The resolution of image is $\delta = 2$ mm . So $\mu_0 = 1, \mu_1 = 55\delta, \mu_2 = 272\delta^2$.

```
omega=zeros(dim1,dim2);
x=40;y=40;
omega(x,y)=1;

Emap=[3 2 1 0];
E=4;F=1;
for i=1:10000
  cur_x = x+ (-1)^binornd(1,0.5);
  cur_y = y+ (-1)^binornd(1,0.5);
  if (image(cur_x,cur_y) >0.1)
    if (omega(cur_x,cur_y) ~=1)
      n_nbr=omega(cur_x-1,cur_y-1)
            +omega(cur_x+1,cur_y+1)
            +omega(cur_x-1,cur_y+1)
            +omega(cur_x+1,cur_y-1);
```

```
      E=E+Emap(n_nbr);
      F=F+1;
    end;
    omega(cur_x,cur_y)=1;
    x=cur_x;
    y=cur_y;
  end;
end;
```

3. *Gaussian fields.* The EC density is given by

$$\rho_d(h) = \mathbb{E}\big[(Y > h)\det(-\ddot{Y}_d)|\dot{Y}_d\big] P(\dot{Y}_d = 0).$$

where dot notation indicates differentiation with respect to the first d components. Computation of this conditional expectation is nontrivial other than Gaussian fields. For zero mean Gaussian field Y,

$$\rho_0 = 1 - \Phi(h),$$

$$\rho_1 = \lambda^{1/2}\frac{e^{-h^2/2}}{\sqrt{2\pi}},$$

$$\rho_2 = \lambda h\frac{e^{-h^2/2}}{\sqrt{2\pi}},$$

$$\rho_3 = \lambda^{3/2}(h^2-1)\frac{e^{-h^2/2}}{\sqrt{2\pi}}$$

where λ will be defined later.

4. *F-fields.* It is defined in Worsley, K.J. (1994). Local maxima and the expected Euler characteristic of excursion sets of $\chi 2$, F and t fields. Advances in Applied Probability, 26:13-42. Let $X_1, \cdots, X_\alpha, Y_1, \cdots, Y_\beta$ be i.i.d. stationary zero mean Gaussian fields. Then F-field with α and β degrees of freedom is given by

$$F(x) = \frac{\sum_{j=1}^{\alpha} X_j^2(x)/\alpha}{\sum_{j=1}^{\beta} Y_j^2(x)/\beta}.$$

To avoid singularity, we need to assume the total degrees of freedom $\alpha + \beta \gg N$ to be fairly larger than the dimension of space. See Worsley (1994) for detail. The EC-density for F-field is given by

$$\begin{aligned}
\rho_0 &= \int_h^\infty \frac{\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}\frac{\alpha}{\beta}\left(\frac{\alpha x}{\beta}\right)^{\frac{(\alpha-2)}{2}}\left(1+\frac{\alpha x}{\beta}\right)^{-\frac{(\alpha+\beta)}{2}} dx,\\
\rho_1 &= \lambda^{1/2}\frac{\Gamma(\frac{\alpha+\beta-1}{2})2^{\frac{1}{2}}}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}\left(\frac{\alpha h}{\beta}\right)^{\frac{(\alpha-1)}{2}}\left(1+\frac{\alpha h}{\beta}\right)^{-\frac{(\alpha+\beta-2)}{2}},\\
\rho_2 &= \lambda\frac{\Gamma(\frac{\alpha+\beta-2}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}\left(\frac{\alpha h}{\beta}\right)^{\frac{(\alpha-2)}{2}}\left(1+\frac{\alpha h}{\beta}\right)^{-\frac{(\alpha+\beta-2)}{2}}\\
&\times \left[(\beta-1)\frac{\alpha h}{\beta} - (\alpha-1)\right].
\end{aligned}$$

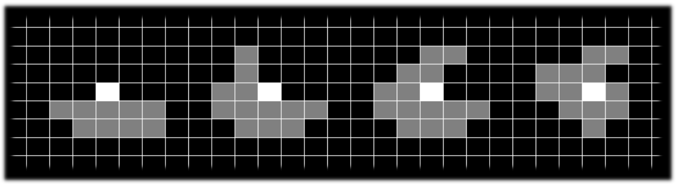

Figure 2: There are only 4 possible configuration of adding additional pixel to a search region Ω. Each case corresponds to adding 3, 2 1 and 0 edges and it has been implemented as a table called `Emap` in the algorithm and figuring out how many edges to add is implemented as a look up table.

5. *Amount of smoothing.* λ measures the smoothness of fields, defined as the variance of the derivative of component of Y. Consider zero mean unit variance field Y, assume that it can be constructed from white noise W by convolving it with Gaussian kernel W, i.e. $Y = K_\sigma * W$. Obviously this generates isotropic fields with covariance function R. Then we can show that

$$\mathbf{Var}\dot{Y} = \lambda I$$

where $\lambda = \frac{4\ln 2}{2\pi\sigma^2}$. Note that $R_Y(x,y) = \int K_\sigma(x-z)K_\sigma(y-z)\,dz$. So

$$\begin{aligned}
\lambda_{ij} &= \mathbb{E}\Big(\frac{\partial Y}{\partial x_i}\frac{\partial Y}{\partial x_j}\Big)\\
&= \frac{\partial^2}{\partial x_i \partial x_j}R_Y(x,x) = \int \frac{\partial K_\sigma(z)}{\partial x_i}\frac{\partial K_\sigma(z)}{\partial x_j}\,dz
\end{aligned}$$

Problem 37. Simplify the above expression and show $\lambda \propto \sigma^{-2}$. If possible generalize the result for the general anisotropic kernel smoothed image $K_H * W$.