Stat 992: Lecture 23
Maxima of random fields.

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1. *Suprema distributions.* Following previous lectures, for zero mean random fields Y, type I-error α was given by

$$\begin{aligned}\alpha &= P\Big(\bigcup_{x\in\Omega}\{Y(x)>h\}\Big)\\ &= P\Big(\sup_{x\in\Omega} Y(x)>h\Big).\end{aligned}$$

So we need to find the distribution of a random variable $\sup_{x\in\Omega} Y(x)$. This is a difficult math problem. Let us compute some simple suprema distributions for Gaussian processes.

Theorem.(*reflection principle*) Consider a Brownian motion $B(x)$ in interval $\Omega = [0,u]$ with $P(B(0)=0)=1$. Then

$$P\Big(\sup_{0\le x\le u} B(x)>h\Big) = \frac{2}{\sqrt{2\pi u}}\int_h^\infty e^{x^2/2u}\,dx.$$

Proof. Note this proof is not rigorous but intuitive. For a rigorous proof, see Freedman's Brownian Motion and Diffusion. Let $M_u = \sup_{0\le x\le u} B(x)$. Define hitting time

$$T_h = \inf\{x : 0\le x, B(x)=h\}.$$

By treating x to be time, T_h is the first time the Brownian motion hits h. Note that

$$P(M_u\ge h) = P(T_h\le u).$$

So by knowing the distribution of T_h, we know the distribution of M_u. Define a reflected process $\widetilde{B}(x)$ to be

$$\widetilde{B}(x) = \Big[\begin{array}{c} B(x) \text{ for } x<T_h,\\ h-[B(x)-h] \text{ for } x\ge T_h.\end{array}$$

Note that B and $\widetilde{B}$ are equal in distribution from symmetry. Then

$$\begin{aligned}P(M_u\ge h) &= P(M_u\ge h, B(u)\ge h)+P(M_u\ge h, B(u)<h)\\ &= P(B(u)\ge h)+P(\widetilde{B}(u)>h)\\ &= 2P(B(u)>h)\end{aligned}$$

2. *Smooth Gaussian process.* Read R.J. Adler's On Excursion Sets, Tube Formulae, and Maxima of Random Fields (1999). The suprima of Brownian motion is simple due to its independent increment properties. Now consider 1-dimensional smooth isotropic Gaussian random process $Y(x), x\in\Omega = [0,1]\subset\mathbb{R}$. Let N_h to be the number of times Y crosses over h from below (*upcrossing*) in $[0,1]$. Then

$$\begin{aligned}P\Big(\sup_{x\in[0,1]} Y(x)>h\Big) &= P(N_h\ge 1 \text{ or } Y(0)>h)\\ &\le P(N_h\ge 1)+P(Y(0)>h)\\ &\le \mathbb{E}N_h + P(Y(0)>h).\end{aligned}$$

Let $\sigma^2 = \mathbb{E}Y^2(x)$. It can be shown that from Rice formula (1945),

$$\mathbb{E}N_h = C_1\exp(h^2/2\sigma^2)$$

for some constant C_1 given in Adler (1999). Also note that $P(Y(0)>h) = 1-\Phi(\frac{h}{\sigma})$ where Φ is the cumulative distribution function of the standard normal. Then

From the well known inequality (Feller, 1968, p.193),

$$(1-\frac{\sigma^2}{h^2})\frac{\sigma}{\sqrt{2\pi}h}e^{-h^2/2\sigma^2}\le 1-\Phi(\frac{h}{\sigma})\le\frac{\sigma}{\sqrt{2\pi}h}e^{-h^2/2\sigma^2}$$

So

$$P\Big(\sup_{x\in[0,1]} Y(x)>h\Big)\le\Big[C_1+\frac{C_2}{h}\Big]e^{h^2/2\sigma^2}.$$

In fact we can show that

$$P\Big(\sup_{x\in[0,1]} Y(x)>h\Big) = \Big[C_1+\frac{C_2}{h}+O(h^{-2})\Big]e^{h^2/2\sigma^2}.$$

Problem 36. Simulate 10,000 smooth zero mean isotropic Gaussian fields in $[0,1]$ in MATLAB and check the validity Rice formula by the Monte-Carlo method. You need to plot $\mathbb{E}N_h$ as a function of h. Also plot $P\Big(\sup_{x\in[0,1]} Y(x)>h\Big)$ as a function of h based on your simulation and check the above asymptotic result is correct.