Stat 992: Lecture 17
Laplace Operator.

Moo K. Chung mchung@stat.wisc.edu

December 11, 2003

1. *Problem statement.* In the previous lecture we showed how to estimate Laplacian in regular grid $\mathbb{Z}^n$. Now we show how to estimate Laplacian in irregular grid such as polygonal surfaces in $\mathbb{R}^2$ for the formulation of diffusion smoothing (Figure 1). The question is how one estimate Laplacian or any other differential operators in the polygonal surface. Assume we have observations Y_i at each point p_i, which is assumed to follow additive model
$$Y_i = \mu(p_i) + \epsilon(p_i), p_i \in \mathbb{R}^2$$
where μ is a smooth continuous function and ϵ zero mean Gaussian random fields. We want to estimate
$$\Delta\mu(p_0) = \frac{\partial^2 \mu}{\partial x^2}\Big|_{p_0} + \frac{\partial^2 \mu}{\partial y^2}\Big|_{p_0}.$$
Unfortunately, the geometry of polygonal surfaces forbid direct application of finite difference scheme. There are a couple of radically different approaches answering this problem (See one of my publication that uses finite element method).

2. *Local polynomial regression.* Let $p_i = (x_i, y_i)$ be the coordinate of vertices of polygons. We estimate the Laplacian at $p_0 = (0,0)$ by fitting a quadratic polynomial of the form
$$\mu(u,v) = \beta_0 + \beta_1 u + \beta_2 v + \beta_3 u^2 + \beta_4 uv + \beta_5 v^2. \quad (1)$$
Let $(u,v) = (x_i, y_i)$ and $\mu(x_i, y_i) = Y_i$ and set up a normal equation, i.e.
$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 y_i + \beta_3 x_i^2 + \beta_4 x_i y_i + \beta_5 y_i^2.$$
Let $Y = (Y_1, \cdots, Y_m)'$, $\beta = (\beta_0, \cdots, \beta_5)'$ and design matrix
$$\mathbb{X} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_m & y_m & x_m^2 & x_my_m & y_m^2 \end{pmatrix}.$$
Then we have the following matrix equation
$$Y = \mathbb{X}\beta.$$

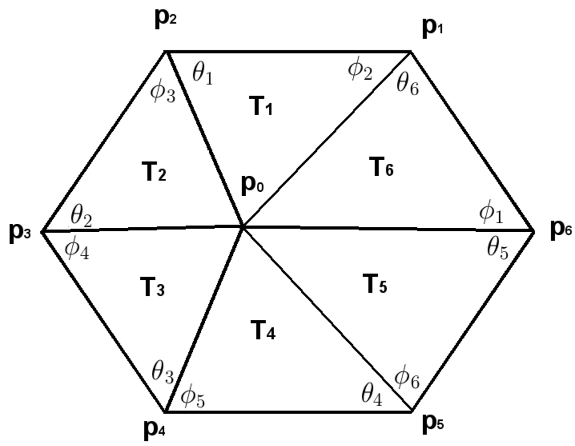

Figure 1: Polygonal mesh in $\mathbb{R}^2$.

The unknown coefficients β_i are estimated by the least-squares approximation such that
$$\hat\beta = (\hat\beta_0, \ldots, \hat\beta_5)' = (\mathbb{X}'\mathbb{X})^{-}\mathbb{X}'Y,$$
where $^-$ denotes generalized inverse, which can be obtained through the singular value decomposition (SVD). Note that $\mathbb{X}'\mathbb{X}$ is nonsingular if $m > 6$.

3. *Moore-Penrose Generalized inverse.* Generalized inverse usually used is that of Moore-Penrose. See C.R. Rao and S.K. Mitra, Generalized inverse of matrices and its applications, Wiley, New Work, 1971; Penrose, R., A Generalized Inverse for Matrices. Proc. Cambridge Phil. Soc. 51, 406-413, 1955. It is usually defined as matrix $\mathbb{X}$ satisfying four conditions
$$\mathbb{X}\mathbb{X}^-\mathbb{X} = \mathbb{X},\ \mathbb{X}^-\mathbb{X}\mathbb{X}^- = \mathbb{X}^-,$$
$$(\mathbb{X}\mathbb{X}^-)' = \mathbb{X}\mathbb{X}^-,\ (\mathbb{X}^-\mathbb{X})' = \mathbb{X}^-\mathbb{X}.$$
Let $\mathbb{X}$ be $m \times p$ matrix with $m \geq p$. Then SVD of $\mathbb{X}$ is
$$\mathbb{X} = UDV',$$
where $U_{m\times p}$ has orthonormal columns, $D_{p\times p} = Diag(d_1, \cdots, d_p)$ is diagonal with non-negative

elements and $V_{p\times p}$ is orthogonal. Let $D^- = Diag(d_1^-, \cdots, d_p^-)$, $d_i^- = 1/d_i$ if $d_i \neq 0$ and $d_i^- = 0$ if $d_i = 0$. Then it can be shown that the Moore-Penrose generalized inverse is given by

$$\mathbb{X}^- = VD^-U'.$$

Then the Laplacian of is

$$\widehat{\Delta\mu}(p_0) = 2\hat{\beta}_3 + 2\hat{\beta}_5.$$

Note that $\hat{\beta}_3$ and $\hat{\beta}_5$ are expressed as the weighted averaging of Y_i's. Hence

$$\widehat{\Delta\mu}(p_0) = \sum_{i=1}^{m} w_i Y_i$$

where weight w_i are equivalent to the the sum of the i-th component of the 3rd and 5th rows of $\mathbb{X}^-$. Note that the weights w_i are functions of the coordinates of vertices $p_1, \cdots, p_m$.

Problem 28. Express weights w_i in terms of x_i's and y_i's. `MAPLE` or `MATHEMATICA` may help you. :)