

Stat 992: Lecture 10

Hilbert Space.

Moo K. Chung `mchung@stat.wisc.edu`

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1. *Hilbert space.* In the previous lecture, we showed how to generate Gaussian fields using simple kernel smoothing. A more general technique is based on orthogonal expansions of the fields.

A Hilbert space \mathcal{H} is a complete inner product space. See F. Riesz and B. Sz.-Nagy's *Functional Analysis* (1955). For every $f, h \in \mathcal{H}$, there is a scalar number $\langle f, h \rangle$ such that mapping $f \rightarrow \langle f, h \rangle$ is a linear functional, i.e. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$. Letting $\alpha = 0$ and $\beta = 0$, $\langle 0, h \rangle = 0$. Further we assume $\langle f, g \rangle = \langle g, f \rangle$ and $\langle f, f \rangle > 0$ if $f \neq 0$. We call such mapping $\langle \cdot, \cdot \rangle$ an inner product. We define metric norm of an element f as $\|f\| = \langle f, f \rangle^{1/2}$. \mathcal{H} is complete if a sequence of elements $f_n \in \mathcal{H}$ with the condition $\|f_n - f_m\| \rightarrow 0$ for $m, n \rightarrow \infty$ has a limit in \mathcal{H} .

A space of functions $L^2(0, 1)$ with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx$$

such that $\|f\| < \infty$ is a Hilbert space. This space is usually denoted by $L^2(0, 1)$. The completeness of $L^2(0, 1)$ space is a standard exercise in real analysis.

2. *Orthogonal expansion.* If $U \subset \mathcal{H}$, the span of U denoted by $Span(U)$ is the collection of all finite linear combinations of elements in U . Further define a closure of $Span(U)$ to be the smallest closed set containing $Span(U)$. For example consider $H = L^2(0, 1)$. $U = \{1, x, x^2, \dots\} \subset H$, $Span(U)$ is the collection of all polynomials while $\overline{Span(U)} = H$. So in this particular example, U is dense in H . The whole idea of orthogonal expansion is to find a relatively smaller countable set U that spans H . We call a set separable if it contains a countable spanning subset. So $L^2(0, 1)$ is separable.

Most Hilbert spaces in practice are separable. So for separable Hilbert space \mathcal{H} , we have a spanning

set $\{u_j\}_0^\infty$. Let $U = \{u_j\}_0^m$ be the subset of independent elements. Then using the Gram-Schmidt orthogonalization, we can construct orthonormal basis $V = \{v_j\}_0^m$ from $U = \{u_j\}_0^m$ such that

$$v_j = \frac{w_j}{\|w_j\|}, w_j = u_j - \sum_{i=0}^{j-1} \langle u_j, e_i \rangle e_i.$$

Then for any element $f \in \mathcal{H}$,

$$f = \sum_{j=0}^{\infty} \alpha_j v_j.$$

The coefficient of the expansion is given by $\alpha_j = \langle f, v_j \rangle$ and it is usually termed generalized fourier coefficients for the obvious reason.

We can do a better approximation than this. Based on finite basis $\{v_j\}_0^m$, we can approximate any element $f \in \mathcal{H}$ in the least-square sense by minimizing

$$\min_{\alpha_j} \left\| f - \sum_{j=0}^m \alpha_j v_j \right\|.$$

The minimum always exists in Hilbert space and this is one reason why the least-squares estimation is a widely method. What happen when we use additional basis v_{m+1} to improve the approximation? Nice thing about orthogonal expansion is that there is no need to estimate α_j again since they are given by $\alpha_j = \langle f, v_j \rangle$ for all $j = 0, \dots, m + 1$.

Problem 20. Construct orthonormal basis $v_j(x), x \in S^1$ that can be used to estimate an integrable function on a unit circle. This can be used in representing functional data on a unit circle, smoothing data via a heat equation etc. After construction, simulate Gaussian random field $\sum_{j=0}^m Z_j v_j(x)$ where i.i.d. $Z_j \sim N(0, \sigma^2)$ in computer. What happens when you add additional basis v_{j+1} ?