Stat 992: Lecture 09
Simulating Gaussian fields.

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1. *White noise.* White noise is defined as a random field whose covariance function is proportional to the Dirac-delta function δ, i.e. $R(x, y) \propto \delta(x - y)$. For instance, me may take $R(x, y) = \lim_{\sigma \to 0} K_\sigma(\|x-y\|)$, the limit of the usual isotropic Gaussian kernel. From a standard mathematical sense, white noise does not exists and it can be characterized via generalized functions.

   One example of white noise is the generalized derivative of Brownian motion (Wiener process) called white Gaussian noise. 1D Wiener process $B(x), x \in \mathbb{R}^+$ is zero mean Gaussian field with covariance function $R_B(x, y) = \min(x, y)$. Based on this definition, we can show $\mathbf{Var}B(x) = x$ by taking $x = y$ in the covariance function and $B(0) = 0$ by letting $x = 0$ in the variance. Increments of Wiener processes in nonoverlapping intervals are i.i.d. Gaussian. Further the paths of Wiener process is continues in the Kolmogorov sense while it is not differentiable. For a different but identical canonical construction of Brownian motion, see Bernt Oksendal's Stochastic Differential Equations (5th edition, 2000). Higher dimensional Brownian motion can be generalized by taking each component of vector fields to be i.i.d. Brownian motion.

   Problem 18. In MATLAB, simulate 1D and 2D version of Brownian motion based on the above properties and check $\mathbf{Var}B(x) = x$ by simulating 10000 Brownian motions. What is the corresponding 2D version ?

2. *Gaussian white noise.* Although the path of Wiener process is not differentiable, we can define the generalized derivative via integration by parts with a smooth function f called a test function in the following way

$$f(x)B(x) = \int_0^x f(y)\frac{dB(y)}{dy}\,dy + \int_0^x \frac{f(y)}{dy}B(y)\,dy.$$

   Taking the expectation on both sides we have

$$\int_0^x f(y)\mathbb{E}\frac{dB(y)}{dy}\,dy = 0.$$

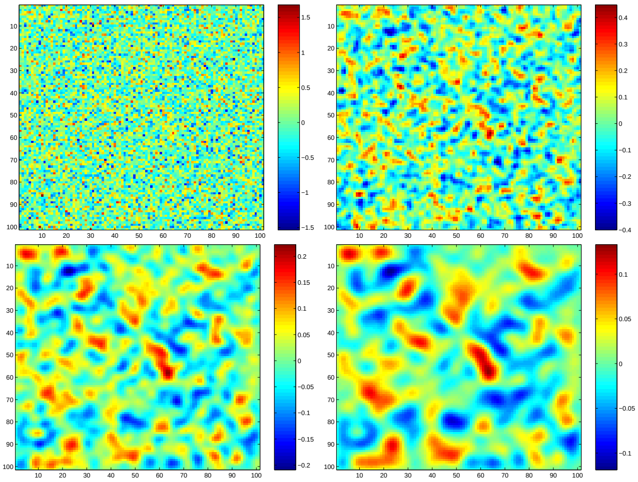

Figure 1: Random fields simulation via iterated Gaussian kernel smoothing with $\sigma = 0.4$. $N(0, 0.4^2)$. White noise, 1, 4 and 9 iterations in sequence.

   It should be true for all smooth f so $\mathbb{E}\frac{dB(y)}{dy} = 0$. Further it can be shown that the covariance function of process $dB(y)/dy \propto \delta(x - y)$.

   Problem 19. Prove this fact and hence showing the generalized derivative of Brownian motion is a white noise. It is usually called a Gaussian white noise.

   The Gaussian white noise can be used to construct smooth Gaussian random fields of the form

$$X(x) = K * W(x) = K * \frac{dB(x)}{dx}.$$

   Since Brownian motion is zero mean Gaussian process, $X(x)$ is obviously zero mean field with the covariance function

$$\begin{aligned} R_X(x, y) &= \mathbb{E}[K * W(x) K * W(y)] \\ &\propto \int K(x - z)K(y - z)\,dz. \end{aligned}$$

   The case $K = K_\sigma$ isotropic Gaussian kernel type has been investigated by D.O. Siegmund and K.J. Worsley with respect to optimal filtering in scale space theory (Annals of Statistics, 23:608-639, 1995).

3. *Numerical Implementation.* In numerical implementation, we use the discrete white Gaussian noise which is simply a Gaussian random variable. Let w be a discrete version of white Gaussian noise given by

$$w(x) = \sum_{i=1}^{m} Z_i \delta(x - x_i)$$

where i.i.d. $Z_i \sim N(0, \sigma_w^2)$. Note that

$$K * w(x) = \sum_{i=1}^{m} Z_i K(x - x_i).$$

Trivially the set of random variables $K * w(y_j)$ forms a multivariate normal at arbitrary points $y_1, \cdots, y_l$. Hence the field $K * w(x)$ is a Gaussian field. The covariance function is given by

$$\begin{aligned} R(x, y) &= \sum_{i,j=1}^{m} \mathbb{E}(Z_i Z_j) K(x - x_i) K(y - x_j) \\ &= \sum_{i=1}^{m} \sigma_w^2 K(x - x_i) K(y - x_i). \end{aligned}$$

As usual we may take K to be a Gaussian kernel. Let us simulate some Gaussian fields. Following the simulation example in lecture 4, the unknown signal is assumed to be $\mu(x, y) = \cos(10x) + \sin(8y), (x, y) \in [0, 1]^2$ and white noise error $w \sim N(0, 0.4^2)$ which gives the left figure in Figure 1.

```
w=normrnd(0,0.4,101,101);
smooth_w=w;
for i=1:10
  smooth_w=conv2(smooth_w,K,'same');
  figure;imagesc(smooth_w);colorbar;
end;
```